

SOLVABILITY OF THE MIXED FORMULATION FOR DARCY–FORCHHEIMER FLOW IN POROUS MEDIA

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Abstract. We consider the mixed formulation of the equations governing Darcy–Forchheimer flow in porous media. We prove existence and uniqueness of a solution for the stationary problem and the existence of a solution for the transient problem.

Résumé. Nous étudions la formulation mixte des équations pour l'écoulement d'un gaz à travers d'un milieu poreux, qu'on suppose régi par la loi de Darcy–Forchheimer. Nous établissons des résultats d'existence et d'unicité pour le problème stationnaire et un résultat d'existence pour le problème transitoire.

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INTRODUCTION

The flow of a gas through a porous medium is governed by the doubly nonlinear parabolic equation

$$\phi(\mathbf{x}) \partial_t \rho(S(\mathbf{x}, t), \mathbf{x}, t) - \operatorname{div} (F(\nabla S(\mathbf{x}, t), \mathbf{x}, t)) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T], \quad (1)$$

where the unknown function $S(= |p|p)$ represents the pressure squared and the nonlinearities ρ and F are defined by

$$\rho(S(\mathbf{x}, t), \mathbf{x}, t) := \gamma(\mathbf{x}, t) \frac{S(\mathbf{x}, t)}{\sqrt{|S(\mathbf{x}, t)|}}, \quad (2)$$

$$F(\nabla S(\mathbf{x}, t), \mathbf{x}, t) := \frac{\sqrt{\alpha(\mathbf{x}, t)^2 + 4\beta(\mathbf{x}, t)|\nabla S(\mathbf{x}, t)|} - \alpha(\mathbf{x}, t)}{2\beta(\mathbf{x}, t)|\nabla S(\mathbf{x}, t)|} \nabla S(\mathbf{x}, t). \quad (3)$$

This equation together with appropriate initial and Neumann boundary conditions has been studied by Amirat [2]. He restricts his considerations to the case, where γ is constant, and shows the existence of a solution using the technique of semi-discretization in time. Under additional regularity conditions on the solution he proves the uniqueness and positivity of this solution. The technique of semi-discretization in time has been used several times to study similar doubly nonlinear parabolic equations. We mention only the articles of Raviart, e.g. [7].

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Equation (1) can be derived from the following system of equations consisting of the Darcy–Forchheimer equation (see e.g. [10])

$$\frac{\mu(\mathbf{x}, t)}{k(\mathbf{x})} \mathbf{u}(\mathbf{x}, t) + \beta_{\text{Fo}}(\mathbf{x}) \rho(\mathbf{x}, t) |\mathbf{u}(\mathbf{x}, t)| \mathbf{u}(\mathbf{x}, t) + \nabla p(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T],$$

the continuity equation

$$\phi(\mathbf{x}) \frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \operatorname{div}(\rho(\mathbf{x}, t) \mathbf{u}(\mathbf{x}, t)) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T]$$

and the ideal gas law as equation of state

$$\rho(\mathbf{x}, t) = \frac{p(\mathbf{x}, t) W(\mathbf{x}, t)}{R_0 \Theta(\mathbf{x}, t)} =: p(\mathbf{x}, t) \gamma(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T].$$

The unknowns here are the pressure p , the density ρ and the volumetric flow rate \mathbf{u} of the gas. Porosity ϕ , permeability k and Forchheimer coefficient β_{Fo} of the porous medium, viscosity μ , molecular weight W and temperature Θ of the gas, and the universal gas constant R_0 are given as well as the source term f . Assuming $\rho > 0$ and introducing new variables $S = |p|p$ and $\mathbf{m} = |\rho| \mathbf{u}$ these equations can be transformed into

$$(\alpha(\mathbf{x}, t) + \beta(\mathbf{x}, t) |\mathbf{m}(\mathbf{x}, t)|) \mathbf{m}(\mathbf{x}, t) + \nabla S(\mathbf{x}, t) = 0, \quad (\mathbf{x}, t) \in \Omega \times [0, T], \quad (4)$$

$$\phi(\mathbf{x}) \partial_t \rho(S(\mathbf{x}, t), \mathbf{x}, t) + \operatorname{div}(\mathbf{m}(\mathbf{x}, t)) = f(\mathbf{x}, t), \quad (\mathbf{x}, t) \in \Omega \times [0, T], \quad (5)$$

where

$$\gamma(\mathbf{x}, t) := \frac{W(\mathbf{x}, t)}{R_0 \Theta(\mathbf{x}, t)}, \quad \alpha(\mathbf{x}, t) := \frac{2 \mu(\mathbf{x}, t)}{\gamma(\mathbf{x}, t) k(\mathbf{x})}, \quad \beta(\mathbf{x}, t) := \frac{2 \beta_{\text{Fo}}(\mathbf{x})}{\gamma(\mathbf{x}, t)},$$

and the equation of state $\rho = \rho(S)$ is defined in (2). Evidently, the Darcy–Forchheimer equation (4) can be resolved to give $\mathbf{m} = F(\nabla S)$ with the nonlinear mapping F defined in (3). Substituting $\mathbf{m} = F(\nabla S)$ into (5) finally yields the parabolic equation (1).

The corresponding stationary problem of system (4–5) together with Neumann boundary conditions has been studied by Fabrie [4] for constant physical parameters. He obtains the existence and uniqueness of a solution $(\mathbf{m}, S) \in (L^3(\Omega))^n \times W^{1,3/2}(\Omega)$ and shows additional regularity properties.

Throughout this article, for $s \in [0, \infty]$ we denote the Lebesgue spaces of s -integrable functions by $L^s(\Omega)$, for $m \geq 0$, $s \in [0, \infty]$ the Sobolev spaces by $W^{m,s}(\Omega)$ and the norm of $W^{m,s}(\Omega)$ by $\|\cdot\|_{m,s,\Omega}$ (cf. [1]). We assume that the domain Ω is bounded and fulfills the uniform C^1 -regularity property. Then the trace operator $\gamma_0 : W^{m,s}(\Omega) \rightarrow W^{m-1/s,s}(\partial\Omega)$ is onto [1, Thm. 7.53]. We denote by $W_0^{m,s}(\Omega)$ the kernel of γ_0 and its dual space by $W^{-m,r}(\Omega) := (W_0^{m,s}(\Omega))'$, where $1/s + 1/r = 1$. Note that for every $s \in [0, \infty]$, $m \geq 0$ the test space $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ is a dense subset of $L^s(\Omega)$ and of $W_0^{m,s}(\Omega)$, and that $\mathcal{D}(\Omega) := \{\Psi|_\Omega \mid \Psi \in \mathcal{D}(\mathbb{R}^n)\}$ is a dense subset of $W^{m,s}(\Omega)$. In addition, let us introduce the generalization $W^s(\operatorname{div}; \Omega)$ of $H(\operatorname{div}; \Omega)$, which is defined in Appendix A. For $s = 3$, this space will turn out to be appropriate for the nonlinearity of the Darcy–Forchheimer law (see Proposition 1.2). Finally, we employ the spaces $C([0, T]; X)$ and $L^s(0, T; X)$ of vector-valued functions, where X is one of the above introduced spaces.

Note that the restriction of [4] prevents from generalization to relevant situations from applications, where the parameters, e.g. the porosity ϕ , vary discontinuously due to composite media or where some of them, e.g. the temperature Θ , are unknowns of a more complex model. Both situations appear in the modelling of combustion in porous media, see [?], which is our final goal. Therefore we study the system (4–5) for Dirichlet boundary conditions and general coefficient functions, imposing only minimal regularity assumptions. We consider the stationary problem in Section 1 and prove the existence and uniqueness of a solution. To this end, we use a regularization of the equations and exploit the monotonicity of the nonlinear mapping F . In Section 2 we investigate the semi-discrete problem after discretization of the time-derivative in (5) and show again the

existence and uniqueness of a solution. Finally, in Section 3, we study the transient problem governed by (4–5) and prove its solvability. Owing to the minor regularity of the solution of the transient problem, we restrict our considerations there to homogeneous boundary conditions.

1. THE STATIONARY PROBLEM

We consider the stationary problem governed by the Darcy–Forchheimer equation and the stationary continuity equation together with Dirichlet boundary conditions:

$$\begin{aligned} (\alpha(\mathbf{x}) + \beta(\mathbf{x})|\mathbf{m}(\mathbf{x})|) \mathbf{m}(\mathbf{x}) + \nabla S(\mathbf{x}) &= 0, & \mathbf{x} \in \Omega, \\ \operatorname{div}(\mathbf{m}(\mathbf{x})) &= f(\mathbf{x}), & \mathbf{x} \in \Omega, \\ S(\mathbf{x}) &= S_b(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{aligned} \quad (6)$$

We require $f \in L^3(\Omega)$, $S_b \in W^{1/3, 3/2}(\partial\Omega)$, $\alpha, \beta \in L^\infty(\Omega)$ and additionally

$$\left. \begin{aligned} 0 &< \underline{\alpha} \leq \alpha(\mathbf{x}) \leq \overline{\alpha} < \infty, \\ 0 &< \underline{\beta} \leq \beta(\mathbf{x}) \leq \overline{\beta} < \infty \end{aligned} \right\} \text{ for almost every } \mathbf{x} \in \Omega.$$

1.1. Mixed formulation of the stationary problem

The mixed formulation of (6) reads as follows: Find $(\mathbf{m}, S) \in W^3(\operatorname{div}; \Omega) \times L^{3/2}(\Omega)$ such that

$$\begin{aligned} \int_{\Omega} (\alpha + \beta|\mathbf{m}|) (\mathbf{m} \cdot \mathbf{v}) \, d\mathbf{x} - \int_{\Omega} \operatorname{div}(\mathbf{v}) S \, d\mathbf{x} &= - \int_{\partial\Omega} S_b(\mathbf{v} \cdot \mathbf{n}) \, d\sigma \quad \text{for all } \mathbf{v} \in W^3(\operatorname{div}; \Omega), \\ \int_{\Omega} \operatorname{div}(\mathbf{m}) q \, d\mathbf{x} &= \int_{\Omega} f q \, d\mathbf{x} \quad \text{for all } q \in L^{3/2}(\Omega). \end{aligned} \quad (7)$$

Next, we introduce continuous linear forms $g : W^3(\operatorname{div}; \Omega) \rightarrow \mathbb{R}$ and $f : L^{3/2}(\Omega) \rightarrow \mathbb{R}$ by means of

$$g(\mathbf{v}) := - \int_{\partial\Omega} S_b(\mathbf{v} \cdot \mathbf{n}) \, d\sigma \quad \text{for } \mathbf{v} \in W^3(\operatorname{div}; \Omega), \quad f(q) := \int_{\Omega} f q \, d\mathbf{x} \quad \text{for } q \in L^{3/2}(\Omega),$$

a bilinear form $b : W^3(\operatorname{div}; \Omega) \times L^{3/2}(\Omega) \rightarrow \mathbb{R}$ and a nonlinear form $a : (L^3(\Omega))^n \times (L^3(\Omega))^n \rightarrow \mathbb{R}$ by means of

$$b(\mathbf{v}, q) := \int_{\Omega} \operatorname{div}(\mathbf{v}) q \, d\mathbf{x} \quad \text{for } \mathbf{v} \in W^3(\operatorname{div}; \Omega), \, q \in L^{3/2}(\Omega), \quad a(\mathbf{u}, \mathbf{v}) := \int_{\Omega} (\alpha + \beta|\mathbf{u}|) (\mathbf{u} \cdot \mathbf{v}) \, d\mathbf{x} \quad \text{for } \mathbf{u}, \mathbf{v} \in (L^3(\Omega))^n.$$

The form b obviously is continuous, the continuity of the form a will be shown in Proposition 1.2. Then we can write the mixed formulation (7) of (6) in the following way: Find $(\mathbf{m}, S) \in W^3(\operatorname{div}; \Omega) \times L^{3/2}(\Omega)$ such that

$$\begin{aligned} a(\mathbf{m}, \mathbf{v}) - b(\mathbf{v}, S) &= g(\mathbf{v}) \quad \text{for all } \mathbf{v} \in W^3(\operatorname{div}; \Omega), \\ b(\mathbf{m}, q) &= f(q) \quad \text{for all } q \in L^{3/2}(\Omega). \end{aligned} \quad (8)$$

In the following let V be one of the spaces $W^3(\operatorname{div}; \Omega)$ or $(L^3(\Omega))^n$. Since a is linear with respect to its second variable, we can define a mapping $A : V \rightarrow V'$ by $\langle A\mathbf{u}, \mathbf{v} \rangle_{V' \times V} = a(\mathbf{u}, \mathbf{v})$, where $\langle \cdot, \cdot \rangle_{V' \times V}$ denotes the dual pairing between V' and V . Using Hölder's inequality we obtain the following bound on $\|A\mathbf{u}\|_{V'}$:

$$\|A\mathbf{u}\|_{V'} \leq C(\overline{\alpha}) \|\mathbf{u}\|_V + C(\overline{\beta}) \|\mathbf{u}\|_V^2, \quad (9)$$

where $C(\overline{\alpha})$ and $C(\overline{\beta})$ are constants depending only on $\overline{\alpha}, \overline{\beta}$ and the domain Ω . Thus $A\mathbf{u}$ is a continuous linear form on V for every $\mathbf{u} \in V$. The proof of (9) is contained in the proof of Proposition 1.2.

The proof of the continuity and monotonicity of the mapping A is based on the following lemma:

Lemma 1.1. *The following inequalities hold for every $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:*

$$|\mathbf{x}|\mathbf{x} - |\mathbf{y}|\mathbf{y}| \leq (|\mathbf{x}| + |\mathbf{y}|) |\mathbf{x} - \mathbf{y}|, \quad (10)$$

$$(|\mathbf{x}|\mathbf{x} - |\mathbf{y}|\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) \geq \frac{1}{2} |\mathbf{x} - \mathbf{y}|^3. \quad (11)$$

The proof of (10) and (11) is elementary. In the two-dimensional case, similar inequalities for more general nonlinearities are derived in [6, Section 5].

In the following we use the notions of [11, Def. 25.2].

Proposition 1.2. *The operator $A : V \rightarrow V'$ is continuous and strictly monotone on $V = W^3(\text{div}; \Omega)$, and continuous, uniformly monotone and coercive on $V = (L^3(\Omega))^n$.*

Proof. To show the continuity of A we consider $\mathbf{u}_1, \mathbf{u}_2 \in (L^3(\Omega))^n$. Applying Hölder's inequality we obtain

$$|\langle A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{v} \rangle_{V \times V'}| \leq \left(C(\bar{\alpha}) \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,3,\Omega} + \bar{\beta} \|\mathbf{u}_1|\mathbf{u}_1 - \mathbf{u}_2|\mathbf{u}_2\|_{0,3/2,\Omega} \right) \|\mathbf{v}\|_{0,3,\Omega} \quad \text{for all } \mathbf{v} \in (L^3(\Omega))^n,$$

which also proves (9) for $V = (L^3(\Omega))^n$ and therefore also for $V = W^3(\text{div}; \Omega)$. Applying inequality (10) and again the Hölder's inequality yields

$$\|A\mathbf{u}_1 - A\mathbf{u}_2\|_{V'} \leq (C(\bar{\alpha}) + C(\bar{\beta}) (\|\mathbf{u}_1\|_{0,3,\Omega} + \|\mathbf{u}_2\|_{0,3,\Omega})) \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,3,\Omega},$$

where $C(\bar{\alpha})$ and $C(\bar{\beta})$ are exactly the same constants as in (9). Using inequality (11) we obtain

$$\langle A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V' \times V} \geq \frac{C(\underline{\beta})}{2} \|\mathbf{u} - \mathbf{v}\|_{0,3,\Omega}^3 \quad \text{for } \mathbf{u}, \mathbf{v} \in (L^3(\Omega))^n, \quad (12)$$

where $C(\underline{\beta})$ depends only on $\underline{\beta}$ and Ω , too. Therefore A is strictly monotone for $V = W^3(\text{div}; \Omega)$, and uniformly monotone (and thus coercive) for $V = (L^3(\Omega))^n$. \square

Remark 1.3. Since A is uniformly monotone for $V = (L^3(\Omega))^n$ we can conclude easily that A is uniformly monotone (and thus coercive) for $V = W_0^3(\text{div}; \Omega) := \{\mathbf{v} \in W^3(\text{div}; \Omega) \mid \text{div}(\mathbf{v}) = 0\}$. Obviously, the solution \mathbf{m} of the homogeneous problem (i.e., $f \equiv 0$) satisfies $\mathbf{m} \in W_0^3(\text{div}; \Omega)$. Therefore, in the homogeneous case, we can extend directly the proof of the unique solvability for the linear problem (cf. [3, Prop. I.1.1 and Thm. I.1.1]) to the nonlinear problem (8). Owing to the uniform monotonicity of A we can use the theorem of Browder and Minty [11, Thm. 26.A] to show that there exists a unique solution $\mathbf{m} \in W_0^3(\text{div}; \Omega)$ of (8). The existence and uniqueness of a solution S then follows exactly like in the proof of [3, Thm. I.1.1].

1.2. Regularization of the stationary problem

For general f the situation is not so simple as depicted in Remark 1.3. We use regularization to show the existence of a solution $(\mathbf{m}, S) \in V \times Q$ to (8), where $V := W^3(\text{div}; \Omega)$ and $Q := L^{3/2}(\Omega)$. For $\varepsilon > 0$ we define nonlinear forms $d_\varepsilon : V \times V \rightarrow \mathbb{R}$ and $c_\varepsilon : Q \times Q \rightarrow \mathbb{R}$ by

$$d_\varepsilon(\mathbf{u}, \mathbf{v}) := \varepsilon \int_{\Omega} |\text{div}(\mathbf{u})| \text{div}(\mathbf{u}) \text{div}(\mathbf{v}) \, d\mathbf{x} \quad \text{for } \mathbf{u}, \mathbf{v} \in W^3(\text{div}; \Omega), \quad c_\varepsilon(p, q) := \varepsilon \int_{\Omega} \frac{p}{\sqrt{|p|}} q \, d\mathbf{x} \quad \text{for } p, q \in L^{3/2}(\Omega).$$

Then the regularized problem is: Find $(\mathbf{m}_\varepsilon, S_\varepsilon) \in V \times Q$ such that

$$\begin{aligned} a(\mathbf{m}_\varepsilon, \mathbf{v}) + d_\varepsilon(\mathbf{m}_\varepsilon, \mathbf{v}) - b(\mathbf{v}, S_\varepsilon) &= g(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V, \\ c_\varepsilon(S_\varepsilon, q) + b(\mathbf{m}_\varepsilon, q) &= f(q) \quad \text{for all } q \in Q. \end{aligned} \quad (13)$$

Analogously to the definition of A we define operators $A_\varepsilon : V \rightarrow V'$ by $\langle A_\varepsilon \mathbf{u}, \mathbf{v} \rangle_{V' \times V} := a(\mathbf{u}, \mathbf{v}) + d_\varepsilon(\mathbf{u}, \mathbf{v})$ and $C_\varepsilon : Q \rightarrow Q'$ by $\langle C_\varepsilon p, q \rangle_{Q' \times Q} := c_\varepsilon(p, q)$. It is evident that $A_\varepsilon \mathbf{u}$ is a linear functional on V for all $\mathbf{u} \in V$, and $C_\varepsilon p$ is a linear functional on Q for all $p \in Q$. Again, continuity of $A_\varepsilon \mathbf{u}$ and $C_\varepsilon p$, resp., follow from the boundedness, which is obtained by means of Hölder's inequality:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + d_\varepsilon(\mathbf{u}, \mathbf{v}) &\leq (C(\bar{\alpha})\|\mathbf{u}\|_{0,3,\Omega} + C(\bar{\beta})\|\mathbf{u}\|_{0,3,\Omega}^2) \|\mathbf{v}\|_{0,3,\Omega} + \varepsilon \|\operatorname{div}(\mathbf{u})\|_{0,3,\Omega}^2 \|\operatorname{div}(\mathbf{v})\|_{0,3,\Omega}, \\ |c_\varepsilon(p, q)| &\leq \varepsilon \|p\|_{0,3/2,\Omega}^{1/2} \|q\|_{0,3/2,\Omega}. \end{aligned}$$

To show that (13) has a solution, we need continuity, coercivity and monotonicity of A_ε and C_ε . For A_ε these properties are consequences of (10-11) again, for C_ε we need an additional lemma:

Lemma 1.4. *The real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto |x|^{-1/2}x$ is strictly monotone and Hölder continuous of order $1/2$ on \mathbb{R} with Hölder constant $\sqrt{2}$, i.e. for all $x, y \in \mathbb{R}$ it holds*

$$|x|^{-1/2}x - |y|^{-1/2}y \leq \sqrt{2}|x - y|^{1/2}. \quad (14)$$

Furthermore

$$\frac{|x - y|^2}{\sqrt{|x|} + \sqrt{|y|}} \leq \left(\frac{x}{\sqrt{|x|}} - \frac{y}{\sqrt{|y|}} \right) (x - y). \quad (15)$$

Proposition 1.5. *For every $\varepsilon > 0$:*

- a) *the operator $A_\varepsilon : V \rightarrow V'$ is continuous, coercive and strictly monotone on V ,*
- b) *the operator $C_\varepsilon : Q \rightarrow Q'$ is continuous, coercive and strictly monotone on Q .*

Proof. Ad a): The continuity of A_ε follows from Proposition 1.2 and the inequality

$$|d_\varepsilon(\mathbf{u}_1, \mathbf{v}) - d_\varepsilon(\mathbf{u}_2, \mathbf{v})| \leq \varepsilon (\|\operatorname{div}(\mathbf{u}_1)\|_{0,3,\Omega} + \|\operatorname{div}(\mathbf{u}_2)\|_{0,3,\Omega}) \|\operatorname{div}(\mathbf{u}_1) - \operatorname{div}(\mathbf{u}_2)\|_{0,3,\Omega} \|\operatorname{div}(\mathbf{v})\|_{0,3,\Omega}$$

for all $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in V$, which is obtained by means of Hölder's inequality and of (10). Furthermore, an application of (11) yields

$$d_\varepsilon(\mathbf{u}, \mathbf{u} - \mathbf{v}) - d_\varepsilon(\mathbf{v}, \mathbf{u} - \mathbf{v}) \geq \frac{\varepsilon}{2} \|\operatorname{div}(\mathbf{u} - \mathbf{v})\|_{0,3,\Omega}^3 \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

Together with (12) this inequality implies the uniform monotonicity of A_ε :

$$\langle A_\varepsilon \mathbf{u} - A_\varepsilon \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V \times V'} \geq \frac{1}{2} \min(C(\underline{\beta}), \varepsilon) \|\mathbf{u} - \mathbf{v}\|_V^3 \quad \text{for all } \mathbf{u}, \mathbf{v} \in V.$$

The coercivity and strict monotonicity of A_ε are direct consequences of the uniform monotonicity.

Ad b): Using Hölder's inequality and (14) we obtain the continuity of C_ε :

$$\|C_\varepsilon p - C_\varepsilon q\|_{Q'} \leq \varepsilon \left\| \frac{p}{\sqrt{|p|}} - \frac{q}{\sqrt{|q|}} \right\|_{0,3,\Omega} \leq \sqrt{2}\varepsilon \|p - q\|_{0,3/2,\Omega}^{1/2} \quad \text{for all } p, q \in Q.$$

The coercivity follows from the equation

$$\langle C_\varepsilon q, q \rangle_{Q' \times Q} = \int_\Omega \varepsilon \frac{q^2}{\sqrt{|q|}} d\mathbf{x} = \varepsilon \|q\|_{0,3/2,\Omega}^{3/2},$$

and strict monotonicity from the strict monotonicity of $x \mapsto |x|^{-1/2}x$:

$$\langle C_\varepsilon p - C_\varepsilon q, p - q \rangle_{Q' \times Q} = \varepsilon \int_\Omega \left(\frac{p}{\sqrt{|p|}} - \frac{q}{\sqrt{|q|}} \right) (p - q) d\mathbf{x} > 0 \quad \text{for all } p, q \in Q \text{ with } p \neq q.$$

□

Now we are in a position to prove:

Proposition 1.6. *For every $\varepsilon > 0$ there is a unique solution $(\mathbf{m}_\varepsilon, S_\varepsilon) \in V \times Q$ of the regularized problem (13).*

Proof. Adding the left hand sides of (13) we obtain the following nonlinear form defined on $V \times Q$:

$$\mathbf{a}_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) + d_\varepsilon(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + c_\varepsilon(p, q) + b(\mathbf{u}, q) \quad \text{for } (\mathbf{u}, p), (\mathbf{v}, q) \in V \times Q$$

and a nonlinear operator $\mathcal{A}_\varepsilon : (V \times Q) \rightarrow (V \times Q)'$ defined by $\langle \mathcal{A}_\varepsilon(\mathbf{u}, p), (\mathbf{v}, q) \rangle_{(V \times Q)' \times (V \times Q)} = \mathbf{a}_\varepsilon((\mathbf{u}, p), (\mathbf{v}, q))$. To study the properties of \mathcal{A}_ε we introduce the continuous linear operator $B : V \rightarrow Q'$ and its adjoint operator $B' : Q \rightarrow V'$, defined by $\langle B\mathbf{v}, q \rangle_{Q \times Q'} = b(\mathbf{v}, q) = \langle B'q, \mathbf{v} \rangle_{V' \times V}$. Then we can write

$$\langle \mathcal{A}_\varepsilon(\mathbf{u}, p), (\mathbf{v}, q) \rangle_{(V \times Q)' \times (V \times Q)} = \langle A_\varepsilon \mathbf{u}, \mathbf{v} \rangle_{V' \times V} - \langle B'p, \mathbf{v} \rangle_{V' \times V} + \langle C_\varepsilon p, q \rangle_{Q' \times Q} + \langle B\mathbf{u}, q \rangle_{Q' \times Q}$$

for $(\mathbf{u}, p), (\mathbf{v}, q) \in V \times Q$. Since A_ε , C_ε , B and B' are continuous, \mathcal{A}_ε is continuous, too. Furthermore, \mathcal{A}_ε is coercive and strictly monotone. This is a straightforward consequence of the corresponding properties of A_ε and C_ε , since the terms containing B or B' cancel each other. For the strict monotonicity this reads

$$\begin{aligned} & \langle \mathcal{A}_\varepsilon(\mathbf{u}, p), (\mathbf{u} - \mathbf{v}, p - q) \rangle_{(V \times Q)' \times (V \times Q)} - \langle \mathcal{A}_\varepsilon(\mathbf{v}, q), (\mathbf{u} - \mathbf{v}, p - q) \rangle_{(V \times Q)' \times (V \times Q)} \\ &= \langle A_\varepsilon \mathbf{u} - A_\varepsilon \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V' \times V} - \langle B'(p - q), \mathbf{u} - \mathbf{v} \rangle_{V' \times V} + \langle C_\varepsilon p - C_\varepsilon q, p - q \rangle_{Q' \times Q} + \langle B(\mathbf{u} - \mathbf{v}), p - q \rangle_{Q' \times Q} \\ &= \langle A_\varepsilon \mathbf{u} - A_\varepsilon \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V' \times V} - \langle B(\mathbf{u} - \mathbf{v}), p - q \rangle_{Q' \times Q} + \langle C_\varepsilon p - C_\varepsilon q, p - q \rangle_{Q' \times Q} + \langle B(\mathbf{u} - \mathbf{v}), p - q \rangle_{Q' \times Q} \\ &= \langle A_\varepsilon \mathbf{u} - A_\varepsilon \mathbf{v}, \mathbf{u} - \mathbf{v} \rangle_{V' \times V} + \langle C_\varepsilon p - C_\varepsilon q, p - q \rangle_{Q' \times Q} > 0, \end{aligned}$$

if $\mathbf{u} \neq \mathbf{v}$ or $p \neq q$. The proof of the coercivity of \mathcal{A}_ε is even more simple.

Thus we can apply the theorem of Browder and Minty [11, Thm. 26.A] to show that for every $\mathbf{f} \in (V \times Q)'$ there exists a unique solution $(\mathbf{m}_\varepsilon, S_\varepsilon) \in V \times Q$ of the operator equation $\mathcal{A}_\varepsilon(\mathbf{m}_\varepsilon, S_\varepsilon) = \mathbf{f}$. In particular, we choose the linear form \mathbf{f} defined by $\mathbf{f}(\mathbf{v}, q) := g(\mathbf{v}) + f(q)$, which arises by adding the right hand sides of (13). Therefore (13) has a unique solution. □

Next, we show that the solution $(\mathbf{m}_\varepsilon, S_\varepsilon)$ is bounded independently of ε .

Proposition 1.7. *There exist constants $\mathcal{K}_\mathbf{m}, \mathcal{K}_S$, independent of ε , such that for sufficiently small $\varepsilon > 0$ the solution $(\mathbf{m}_\varepsilon, S_\varepsilon)$ of (13) satisfies the following estimates:*

$$\|\mathbf{m}_\varepsilon\|_V \leq \mathcal{K}_\mathbf{m}, \quad \|S_\varepsilon\|_Q \leq \mathcal{K}_S. \quad (16)$$

Proof. We begin with a bound for the norm of $\text{div}(\mathbf{m}_\varepsilon)$. Using the second equation of (13) we obtain

$$\|\text{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega} = \|\text{div}(\mathbf{m}_\varepsilon)\|_{Q'} = \sup_{q \in Q} \frac{|b(\mathbf{m}_\varepsilon, q)|}{\|q\|_Q} = \sup_{q \in Q} \frac{|f(q) - c_\varepsilon(S_\varepsilon, q)|}{\|q\|_Q} \leq \|f\|_{0,3,\Omega} + \varepsilon \|S_\varepsilon\|_Q^{1/2}.$$

The estimation of $\|\mathbf{m}_\varepsilon\|_{0,3,\Omega}$ is based on the first equation in (13):

$$\begin{aligned} C(\underline{\beta}) \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^3 &\leq \int_\Omega \beta |\mathbf{m}_\varepsilon| (\mathbf{m}_\varepsilon \cdot \mathbf{m}_\varepsilon) \, d\mathbf{x} \leq a(\mathbf{m}_\varepsilon, \mathbf{m}_\varepsilon) + d_\varepsilon(\mathbf{m}_\varepsilon, \mathbf{m}_\varepsilon) = g(\mathbf{m}_\varepsilon) + b(\mathbf{m}_\varepsilon, S_\varepsilon) \\ &\leq \|g\|_{V'} \|\mathbf{m}_\varepsilon\|_{0,3,\Omega} + (\|g\|_{V'} + \|S_\varepsilon\|_Q) \|\text{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}. \end{aligned}$$

Together with the estimate for $\|\text{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}$ above this yields

$$\|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^3 \leq \frac{1}{C(\underline{\beta})} \left(\|g\|_{V'} \|\mathbf{m}_\varepsilon\|_{0,3,\Omega} + \|g\|_{V'} \|f\|_{0,3,\Omega} + \varepsilon \|g\|_{V'} \|S_\varepsilon\|_Q^{1/2} + \|f\|_{0,3,\Omega} \|S_\varepsilon\|_Q + \varepsilon \|S_\varepsilon\|_Q^{3/2} \right). \quad (17)$$

To bound S_ε we employ the inf-sup condition (38). Together with the first equation in (13) and the above estimate for $\|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}$ we obtain

$$\begin{aligned} \theta \|S_\varepsilon\|_Q &\leq \sup_{\mathbf{v} \in V} \frac{b(\mathbf{v}, S_\varepsilon)}{\|\mathbf{v}\|_V} = \sup_{\mathbf{v} \in V} \frac{a(\mathbf{m}_\varepsilon, \mathbf{v}) + d_\varepsilon(\mathbf{m}_\varepsilon, \mathbf{v}) - g(\mathbf{v})}{\|\mathbf{v}\|_V} \\ &\leq \|A\mathbf{m}_\varepsilon\|_{V'} + \varepsilon \|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}^2 + \|g\|_{V'} \leq \|A\mathbf{m}_\varepsilon\|_{V'} + \|g\|_{V'} + \varepsilon \left(\|f\|_{0,3,\Omega} + \varepsilon \|S_\varepsilon\|_Q^{1/2} \right)^2 \end{aligned}$$

for some constant $\theta > 0$. Thus for sufficiently small ε ($\varepsilon < \theta^{1/3}$ is enough) it holds

$$\|S_\varepsilon\|_Q \leq \frac{1}{\theta - \varepsilon^3} \left(\|A\mathbf{m}_\varepsilon\|_{V'} + \|g\|_{V'} + \varepsilon \|f\|_{0,3,\Omega}^2 + 2\varepsilon^2 \|f\|_{0,3,\Omega} \|S_\varepsilon\|_Q^{1/2} \right),$$

such that

$$\|S_\varepsilon\|_Q^{1/2} \leq \left(\frac{2\varepsilon^2}{\theta - \varepsilon^3} \|f\|_{0,3,\Omega} + \left(\frac{1}{\theta - \varepsilon^3} \left(\|A\mathbf{m}_\varepsilon\|_{V'} + \|g\|_{V'} + \varepsilon \|f\|_{0,3,\Omega}^2 \right) \right)^{1/2} \right).$$

An application of the estimate $\|A\mathbf{m}_\varepsilon\|_{V'} \leq C(\bar{\alpha}) \|\mathbf{m}_\varepsilon\|_{0,3,\Omega} + C(\bar{\beta}) \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^2$ finally yields

$$\|S_\varepsilon\|_Q^{1/2} \leq \kappa_0 + \kappa_1 \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^{1/2} + \kappa_2 \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}, \quad (18)$$

where the coefficients κ_i are bounded, independently of \mathbf{m}_ε and S_ε , for sufficiently small ε , e.g. $\varepsilon < \theta^{1/3}/2$,

$$\kappa_0 := \frac{2\varepsilon^2}{\theta - \varepsilon^3} \|f\|_{0,3,\Omega} + \left(\frac{1}{\theta - \varepsilon^3} (\|g\|_{V'} + \varepsilon \|f\|_{0,3,\Omega}^2) \right)^{1/2}, \quad \kappa_1 := \left(\frac{C(\bar{\alpha})}{\theta - \varepsilon^3} \right)^{1/2}, \quad \kappa_2 := \left(\frac{C(\bar{\beta})}{\theta - \varepsilon^3} \right)^{1/2}.$$

Inserting (18) into (17) we obtain after some calculations the inequality

$$\left(1 - \varepsilon \frac{\kappa_2^2}{\beta c_\ell^3} \right) \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^3 \leq \sum_{i=0}^5 \lambda_i \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^{i/2},$$

where the λ_i ($i = 0, \dots, 5$) are independent of $\|\mathbf{m}_\varepsilon\|_{0,3,\Omega}$ and bounded for sufficiently small $\varepsilon < \bar{\varepsilon} \leq \theta^{1/3}/2$. Thus there exists a constant $\mathcal{K}_1 < \infty$, independent of ε , such that $\|\mathbf{m}_\varepsilon\|_{0,3,\Omega} \leq \mathcal{K}_1$ for $\varepsilon \leq \bar{\varepsilon}$. Inserting this estimate into (18) we obtain the bound for $\|S_\varepsilon\|_Q$ and using the above estimate for $\|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}$ finally yields the bound for $\|\mathbf{m}_\varepsilon\|_V$. \square

1.3. Solvability of the stationary problem (8)

Theorem 1.8. *The mixed formulation (8) of the stationary problem (6) possesses a unique solution $(\mathbf{m}, S) \in W^3(\operatorname{div}; \Omega) \times L^{3/2}(\Omega)$.*

Proof. Analogously to the definition of \mathbf{a}_ε we add the left hand sides of (8) and obtain the nonlinear form \mathbf{a} defined by $\mathbf{a}((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + b(\mathbf{u}, q)$ and the nonlinear operator $\mathcal{A} : (V \times Q) \rightarrow (V \times Q)'$ defined by $\langle \mathcal{A}(\mathbf{u}, p), (\mathbf{v}, q) \rangle_{(V \times Q)' \times (V \times Q)} = \mathbf{a}((\mathbf{u}, p), (\mathbf{v}, q))$. Setting $\varepsilon = 1/n$ let (\mathbf{m}_n, S_n) be the unique solution of the regularized problem (13). Since $((\mathbf{m}_n, S_n))_{n \in \mathbb{N}}$ is a bounded sequence in $V \times Q$, there exists a weakly

convergent subsequence, again denoted by $((\mathbf{m}_n, S_n))_{n \in \mathbb{N}}$, with (weak) limit $(\mathbf{m}, S) \in V \times Q$. As

$$\begin{aligned} \|\mathcal{A}(\mathbf{m}_n, S_n) - \mathbf{f}\|_{(V \times Q)'} &= \sup_{0 \neq (\mathbf{v}, q) \in V \times Q} \frac{|\mathbf{a}((\mathbf{m}_n, S_n), (\mathbf{v}, q)) - \mathbf{f}(\mathbf{v}, q)|}{\|(\mathbf{v}, q)\|_{V \times Q}} \\ &= \sup_{(\mathbf{v}, q)} \frac{|\mathbf{a}_{1/n}((\mathbf{m}_n, S_n), (\mathbf{v}, q)) - d_{1/n}(\mathbf{m}_n, \mathbf{v}) - c_{1/n}(S_n, q) - \mathbf{f}(\mathbf{v}, q)|}{\|(\mathbf{v}, q)\|_{V \times Q}} \\ &\leq \frac{1}{n} \left(\|\operatorname{div}(\mathbf{m}_n)\|_{0,3,\Omega}^2 + \|S_n\|_Q^{1/2} \right) \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

the sequence $(\mathcal{A}(\mathbf{m}_n, S_n))_{n \in \mathbb{N}}$ converges strongly in V' to \mathbf{f} defined by $\mathbf{f}(\mathbf{v}, q) := g(\mathbf{v}) + f(q)$. Thus we can conclude that $\mathcal{A}(\mathbf{m}, S) = \mathbf{f}$ in $(V \times Q)'$ (see e.g. [11, p. 474]), i.e., (\mathbf{m}, S) is a solution of (8).

To show uniqueness, we consider two solutions (\mathbf{m}_1, S_1) and (\mathbf{m}_2, S_2) of (8). Using the test functions $\mathbf{v} = \mathbf{m}_1 - \mathbf{m}_2$ and $q = S_1 - S_2$ we obtain

$$\begin{aligned} a(\mathbf{m}_1, \mathbf{m}_1 - \mathbf{m}_2) - a(\mathbf{m}_2, \mathbf{m}_1 - \mathbf{m}_2) - b(\mathbf{m}_1 - \mathbf{m}_2, S_1) + b(\mathbf{m}_1 - \mathbf{m}_2, S_2) &= 0 \\ b(\mathbf{m}_1, S_1 - S_2) - b(\mathbf{m}_2, S_1 - S_2) &= 0. \end{aligned}$$

Adding these equations yields

$$0 = a(\mathbf{m}_1, \mathbf{m}_1 - \mathbf{m}_2) - a(\mathbf{m}_2, \mathbf{m}_1 - \mathbf{m}_2) = \langle A\mathbf{m}_1 - A\mathbf{m}_2, \mathbf{m}_1 - \mathbf{m}_2 \rangle_{(V \times Q)' \times (V \times Q)}.$$

Since A is strictly monotone it follows $\mathbf{m}_1 - \mathbf{m}_2 = 0$.

If $\mathbf{m} \in V$ is given, $S \in Q$ is defined as solution of the variational equation $b(\mathbf{v}, S) = g(\mathbf{v}) - a(\mathbf{m}, \mathbf{v})$ for all $\mathbf{v} \in V$. Therefore the uniqueness of S is a direct consequence of the injectivity of the operator $B' : Q \rightarrow V'$, cf. [3, §II, Rem. 1.6]. \square

2. THE SEMI-DISCRETE PROBLEM

We return to the transient problem governed by (4) and (5). We discretize (5) in time using the implicit Euler method. This yields not only a method to solve the transient problem numerically, but also an approach to prove its solvability, the technique of semi-discretization.

We define a partition $0 = t^0 < t^1 < \dots < t^K = T$ of the segment $(0, T)$ into K intervals of constant length $\Delta t = T/K$, i.e., $t_k = k\Delta t$ for $k = 0, \dots, K$. In the following for $k = 0, \dots, K$ we use the denotations $S^k := S(\cdot, t^k)$ and $\mathbf{m}^k := \mathbf{m}(\cdot, t^k)$ for the unknown solutions and, analogously defined, α^k , β^k and γ^k for the coefficient functions, S_b^k for the boundary conditions and f^k for the source term. The initial condition $S(\cdot, t^0) = S^0(\cdot) \in W_0^{1,3/2}(\Omega)$ is given.

Using the equation of state $\rho = \rho(S)$ defined in (2), the discretization in time of the continuity equation (5) with the implicit Euler method yields for each $k \in \{1, \dots, K\}$

$$\begin{aligned} (\alpha^k(\mathbf{x}) + \beta^k(\mathbf{x})|\mathbf{m}^k(\mathbf{x})|) \mathbf{m}^k(\mathbf{x}) + \nabla S^k(\mathbf{x}) &= 0, & \mathbf{x} \in \Omega, \\ \frac{\phi(\mathbf{x})}{\Delta t} \left(\gamma^k(\mathbf{x}) \frac{S^k(\mathbf{x})}{\sqrt{|S^k(\mathbf{x})|}} - \gamma^{k-1}(\mathbf{x}) \frac{S^{k-1}(\mathbf{x})}{\sqrt{|S^{k-1}(\mathbf{x})|}} \right) + \operatorname{div}(\mathbf{m}^k(\mathbf{x})) &= f^k(\mathbf{x}), & \mathbf{x} \in \Omega, \\ S(\mathbf{x}) &= S_b^k(\mathbf{x}), & \mathbf{x} \in \partial\Omega. \end{aligned} \tag{19}$$

Note that for each $k \in \{1, \dots, K\}$ the function S^{k-1} is known.

For each $k \in \{1, \dots, K\}$ we require $f^k \in L^3(\Omega)$, $S_b^k \in W^{1/3,3/2}(\partial\Omega)$, $\phi, \alpha^k, \beta^k, \gamma^k \in L^\infty(\Omega)$ and additionally

$$\left. \begin{aligned} 0 < \underline{\phi} &\leq \phi(\mathbf{x}) \leq \overline{\phi} < \infty, \\ 0 < \underline{\alpha} &\leq \alpha^k(\mathbf{x}) \leq \overline{\alpha} < \infty, \\ 0 < \underline{\beta} &\leq \beta^k(\mathbf{x}) \leq \overline{\beta} < \infty, \\ 0 < \underline{\gamma} &\leq \gamma^k(\mathbf{x}) \leq \overline{\gamma} < \infty \end{aligned} \right\} \text{ for almost every } \mathbf{x} \in \Omega.$$

2.1. Mixed formulation of the semi-discrete problem

We continue to use the spaces $V = W^3(\text{div}; \Omega)$ and $Q = L^{3/2}(\Omega)$. Then the variational formulation reads: Find $(\mathbf{m}^k, S^k) \in V \times Q$ such that

$$\begin{aligned} \int_{\Omega} (\alpha^k + \beta^k |\mathbf{m}^k|) (\mathbf{m}^k \cdot \mathbf{v}) \, d\mathbf{x} - \int_{\Omega} \text{div}(\mathbf{v}) S^k \, d\mathbf{x} &= - \int_{\partial\Omega} S_b^k (\mathbf{v} \cdot \mathbf{n}) \, d\sigma && \text{for all } \mathbf{v} \in V, \\ \int_{\Omega} \frac{\phi \gamma^k}{\Delta t} \frac{S^k}{\sqrt{|S^k|}} q \, d\mathbf{x} + \int_{\Omega} \text{div}(\mathbf{m}^k) q \, d\mathbf{x} &= \int_{\Omega} f^k q \, d\mathbf{x} + \int_{\Omega} \frac{\phi \gamma^{k-1}}{\Delta t} \frac{S^{k-1}}{\sqrt{|S^{k-1}|}} q \, d\mathbf{x} && \text{for all } q \in Q. \end{aligned} \quad (20)$$

We introduce additional nonlinear forms c^k on $Q \times Q$ defined by

$$c^k(p, q) := \int_{\Omega} \frac{\phi \gamma^k}{\Delta t} \frac{p}{\sqrt{|p|}} q \, d\mathbf{x}$$

and nonlinear operators $C^k : Q \rightarrow Q'$ by $\langle C^k p, q \rangle_{Q' \times Q} := c^k(p, q)$. Again, it is evident that $C^k p$ is a linear mapping on Q for all $p \in Q$. The continuity of $C^k p$ is equivalent to its boundedness, which in turn is a consequence of Hölder's inequality and the boundedness of ϕ and γ :

$$|c^k(p, q)| \leq \frac{\overline{\phi} \overline{\gamma}}{\Delta t} \|p\|_{0,3/2,\Omega}^{1/2} \|q\|_{0,3/2,\Omega}.$$

In the same manner as in the proof of Proposition 1.5 b) we obtain the continuity, coercivity and monotonicity of C^k .

Proposition 2.1. *The operators $C^k : Q \rightarrow Q'$ are continuous, coercive and strictly monotone on Q .*

We use a , b and g as defined in Section 1, where $a = a^k$ and $g = g^k$ depend on k , because α , β and the boundary condition S_b may change in time. Then we can write the mixed formulation (20) of (19) in the following way: Find $(\mathbf{m}^k, S^k) \in V \times Q$, such that

$$\begin{aligned} a^k(\mathbf{m}^k, \mathbf{v}) - b(\mathbf{v}, S^k) &= g^k(\mathbf{v}) && \text{for all } \mathbf{v} \in V, \\ c^k(S^k, q) + b(\mathbf{m}^k, q) &= \tilde{f}^k(q) && \text{for all } q \in Q. \end{aligned} \quad (21)$$

Here $\tilde{f}^k \in Q'$ for $k = 1, \dots, K$ is defined by

$$\tilde{f}^k(q) := \int_{\Omega} \left(f^k + \frac{\phi \gamma^{k-1}}{\Delta t} \frac{S^{k-1}}{\sqrt{|S^{k-1}|}} \right) q \, d\mathbf{x}.$$

For the remainder of this section we restrict our considerations to a fixed time step k . Thus we can omit the superscript k .

2.2. Regularization of the semi-discrete problem

We use the technique of regularization again. Thus we consider, instead of (21), the following regularized problem for $\varepsilon > 0$: Find $(\mathbf{m}_\varepsilon, S_\varepsilon) \in V \times Q$ such that

$$\begin{aligned} a(\mathbf{m}_\varepsilon, \mathbf{v}) + d_\varepsilon(\mathbf{m}_\varepsilon, \mathbf{v}) - b(\mathbf{v}, S_\varepsilon) &= g(\mathbf{v}) \quad \text{for all } \mathbf{v} \in V, \\ c(S_\varepsilon, q) + b(\mathbf{m}_\varepsilon, q) &= \tilde{f}(q) \quad \text{for all } q \in Q. \end{aligned} \quad (22)$$

Here $d_\varepsilon(\mathbf{u}, \mathbf{v}) := \varepsilon \int_\Omega |\operatorname{div}(\mathbf{u})| \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) \, d\mathbf{x}$ is defined as in Section 1.

In the same manner as Proposition 1.6 we obtain:

Proposition 2.2. *For every $\varepsilon > 0$ there exists a unique solution $(\mathbf{m}_\varepsilon, S_\varepsilon) \in V \times Q$ of the regularized semi-discrete problem (22).*

Next, we show that the solution $(\mathbf{m}_\varepsilon, S_\varepsilon)$ of (22) is bounded independently of ε , too. Since we added in (22) only one regularizing term d_ε , we can use different techniques for the estimation of \mathbf{m}_ε and S_ε . In particular we obtain estimates that hold for every $\varepsilon > 0$:

Proposition 2.3. *There exist constants $\mathcal{K}_\mathbf{m}, \mathcal{K}_S$, independent of ε , such that the solution $(\mathbf{m}_\varepsilon, S_\varepsilon)$ of (22) satisfies the following estimates:*

$$\|\mathbf{m}_\varepsilon\|_V \leq \mathcal{K}_\mathbf{m}, \quad \|S_\varepsilon\|_Q \leq \mathcal{K}_S. \quad (23)$$

Proof. As in the proof of Proposition 1.7 we begin with an estimate for the norm of $\operatorname{div}(\mathbf{m}_\varepsilon)$:

$$\|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega} \leq \|\tilde{f}\|_{Q'} + \frac{\bar{\phi}\bar{\gamma}}{\Delta t} \|S_\varepsilon\|_Q^{1/2}.$$

The estimation of $\|\mathbf{m}_\varepsilon\|_{0,3,\Omega}$ uses the following inequality, established in the proof of Proposition 1.7,

$$C(\underline{\beta}) \|\mathbf{m}_\varepsilon\|_{0,3,\Omega}^3 \leq \|g\|_{V'} \|\mathbf{m}_\varepsilon\|_{0,3,\Omega} + (\|g\|_{V'} + \|S_\varepsilon\|_Q) \|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}$$

to derive

$$\|\mathbf{m}_\varepsilon\|_{0,3,\Omega} \leq \left(\frac{1}{C(\underline{\beta})} \|g\|_{V'} \right)^{1/2} + \left(\frac{1}{C(\underline{\beta})} (\|g\|_{V'} + \|S_\varepsilon\|_Q) \|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega} \right)^{1/3}.$$

Together with the estimate for $\|\operatorname{div}(\mathbf{m}_\varepsilon)\|_{0,3,\Omega}$ we obtain the following bound for $\|\mathbf{m}_\varepsilon\|_V$:

$$\|\mathbf{m}_\varepsilon\|_V \leq \kappa_1 + \kappa_2 \|S_\varepsilon\|_Q^{1/2},$$

where the constants κ_1 and κ_2 are independent of ε and $\|S_\varepsilon\|_Q$. To derive an estimation for $\|S_\varepsilon\|_Q$, we use in (22) the test functions $\mathbf{v} = \mathbf{m}_\varepsilon$ and $q = S_\varepsilon$ and add the resulting equations. Since b is a bilinear form, we obtain the inequality

$$c(S_\varepsilon, S_\varepsilon) \leq a(\mathbf{m}_\varepsilon, \mathbf{m}_\varepsilon) + d_\varepsilon(\mathbf{m}_\varepsilon, \mathbf{m}_\varepsilon) + c(S_\varepsilon, S_\varepsilon) = g(\mathbf{m}_\varepsilon) + \tilde{f}(S_\varepsilon).$$

Using the coercivity of C and the bound for $\|\mathbf{m}_\varepsilon\|_V$ derived above we can therefore conclude

$$\begin{aligned} \frac{\phi\gamma}{\Delta t} \|S_\varepsilon\|_Q^{3/2} &\leq c(S_\varepsilon, S_\varepsilon) \leq g(\mathbf{m}_\varepsilon) + \tilde{f}(S_\varepsilon) \leq \|g\|_{V'} \|\mathbf{m}_\varepsilon\|_V + \|\tilde{f}\|_{Q'} \|S_\varepsilon\|_Q \\ &\leq \|g\|_{V'} \left(\kappa_1 + \kappa_2 \|S_\varepsilon\|_Q^{1/2} \right) + \|\tilde{f}\|_{Q'} \|S_\varepsilon\|_Q. \end{aligned}$$

This yields the existence of a bound \mathcal{K}_S for $\|S_\varepsilon\|_Q$. □

2.3. Solvability of the semi-discrete problem (21)

Again, we consider the limit $\varepsilon \rightarrow 0$ and obtain in the same manner as in Section 1 the existence of a solution of the semi-discrete problem (21). The proof of the uniqueness differs from the proof of Theorem 1.8.

Theorem 2.4. *The mixed formulation (21) of the semi-discrete problem (19) possesses a unique solution $(\mathbf{m}, S) \in W^3(\text{div}; \Omega) \times L^{3/2}(\Omega)$.*

Proof. Like in the proof of Theorem 1.8 we add both equations in (21) and obtain the nonlinear form a , defined on $(V \times Q) \times (V \times Q)$, and the linear form $\mathbf{f} \in (V \times Q)'$, defined by

$$\mathbf{a}((\mathbf{u}, p), (\mathbf{v}, q)) := a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) + c(p, q) + b(\mathbf{u}, q) \quad , \quad \mathbf{f}(\mathbf{v}, q) := g(\mathbf{v}) + \tilde{f}(q) \, .$$

Again, the operator $\mathcal{A} : V \times Q \rightarrow (V \times Q)'$ is defined by $\langle \mathcal{A}(\mathbf{u}, p), (\mathbf{v}, q) \rangle_{(V \times Q)' \times (V \times Q)} = \mathbf{a}((\mathbf{u}, p), (\mathbf{v}, q))$. Choosing $\varepsilon = 1/n$ for $n \in \mathbb{N}$ we obtain a sequence of unique solutions (\mathbf{m}_n, S_n) of the regularized problems (22). Owing to Proposition 2.3 the sequence $((\mathbf{m}_n, S_n))_{n \in \mathbb{N}}$ is bounded in $V \times Q$. Thus there is a weakly convergent subsequence, again denoted by $((\mathbf{m}_n, S_n))_{n \in \mathbb{N}}$, which converges to $(\mathbf{m}, S) \in V \times Q$. In the same manner as in the proof of Theorem 1.8 we obtain the identity $\mathcal{A}(\mathbf{m}, S) = \mathbf{f}$ in $(V \times Q)'$, i.e., (\mathbf{m}, S) is a solution of the semi-discrete mixed formulation (21).

Uniqueness of the solution $(\mathbf{m}, S) \in V \times Q$ follows from the strict monotonicity of \mathcal{A} , which, in turn, is a consequence of the strict monotonicity of A and C . \square

3. THE TRANSIENT PROBLEM

Finally, we address the continuous transient problem. We restrict our considerations here to the case of homogeneous Dirichlet boundary conditions. Due to the lack of regularity of the solution \mathbf{m} , it is not possible to handle more general boundary conditions as in the former sections. Thus we consider the following problem:

$$\begin{aligned} (\alpha(\mathbf{x}, t) + \beta(\mathbf{x}, t)|\mathbf{m}(\mathbf{x}, t)|) \mathbf{m}(\mathbf{x}, t) + \nabla S(\mathbf{x}, t) &= 0 \, , & (\mathbf{x}, t) \in \Omega \times [0, T] \, , \\ \phi(\mathbf{x}) \frac{\partial \rho(S(\mathbf{x}, t), \mathbf{x}, t)}{\partial t} + \text{div}(\mathbf{m}(\mathbf{x}, t)) &= f(\mathbf{x}, t) \, , & (\mathbf{x}, t) \in \Omega \times [0, T] \, , \\ S(\mathbf{x}, t) &= 0 \, , & (\mathbf{x}, t) \in \partial\Omega \times [0, T] \, , \\ S(\mathbf{x}, 0) &= S^0(\mathbf{x}) \, , & \mathbf{x} \in \Omega \, . \end{aligned} \tag{24}$$

Again, we require that $S^0 \in W_0^{1,3/2}(\Omega)$, $\phi \in L^\infty(\Omega)$ with lower and upper bound $0 < \underline{\phi} \leq \phi(\mathbf{x}) \leq \overline{\phi} < \infty$ for almost every $\mathbf{x} \in \Omega$. For every $t \in [0, T]$ the time-varying coefficient functions have to satisfy the following assumptions: $f(\cdot, t) \in L^3(\Omega)$ and $\alpha(\cdot, t), \beta(\cdot, t), \gamma(\cdot, t) \in L^\infty(\Omega)$ with upper and lower bounds

$$\left. \begin{aligned} 0 < \underline{\alpha} \leq \alpha(\mathbf{x}, t) \leq \overline{\alpha} < \infty \, , \\ 0 < \underline{\beta} \leq \beta(\mathbf{x}, t) \leq \overline{\beta} < \infty \, , \\ 0 < \underline{\gamma} \leq \gamma(\mathbf{x}, t) \leq \overline{\gamma} < \infty \end{aligned} \right\} \text{ for almost every } \mathbf{x} \in \Omega \text{ and every } t \in [0, T] \, .$$

Furthermore, we require these coefficient functions to be Lipschitz continuous in time, i.e., there exist constants $L(\alpha)$, $L(\beta)$, $L(\gamma)$ and $L(f)$ such that for every $0 \leq t_1 \leq t_2 \leq T$:

$$\begin{aligned} \|\alpha(t_1) - \alpha(t_2)\|_{0,\infty,\Omega} &\leq L(\alpha) |t_1 - t_2| \, , \\ \|\beta(t_1) - \beta(t_2)\|_{0,\infty,\Omega} &\leq L(\beta) |t_1 - t_2| \, , \quad \text{and} \quad \|f(t_1) - f(t_2)\|_{0,3,\Omega} \leq L(f) |t_1 - t_2| \, . \\ \|\gamma(t_1) - \gamma(t_2)\|_{0,\infty,\Omega} &\leq L(\gamma) |t_1 - t_2| \end{aligned}$$

3.1. A priori estimates for the solutions of the semi-discrete problems

As mentioned above we use the technique of semi-discretization in time to show the existence of solutions of the transient problem (24). One important step has been done in Section 2: The existence and uniqueness of the solutions to the semi-discrete problems has been established. In the next step, we have to consider the limit $\Delta t \rightarrow 0$ (or $K \rightarrow \infty$). Similar to the regularization technique employed in the last two sections, we therefore have to provide a priori estimates for the solutions of the semi-discrete problems, which are independent of Δt . The bounds $\mathcal{K}_{\mathbf{m}}$ and \mathcal{K}_S of Proposition 2.3 do not fulfill this requirement. Thus we investigate the semi-discrete problem (20) for homogeneous Dirichlet boundary conditions. In a slightly different notation this problem reads:

$$\begin{aligned} a^k(\mathbf{m}^k, \mathbf{v}) - b(\mathbf{v}, S^k) &= 0 & \text{for all } \mathbf{v} \in V, \\ \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q \, d\mathbf{x} + b(\mathbf{m}^k, q) &= f^k(q) & \text{for all } q \in Q, \end{aligned} \quad (25)$$

where $\rho^k(S^k) := \gamma^k S^k / \sqrt{|S^k|}$.

Lemma 3.1. *For sufficiently small $\Delta t > 0$ there exists a constant C_S , independent of Δt (and of K), such that*

$$\|S^k\|_{0,3/2,\Omega} \leq C_S \quad \text{for all } 0 \leq k \leq K. \quad (26)$$

Proof. Choosing $\mathbf{v} = \mathbf{m}^k$ and $q = S^k$ in (25) and adding the resulting equations yields

$$a^k(\mathbf{m}^k, \mathbf{m}^k) + \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k \, d\mathbf{x} = f^k(S^k). \quad (27)$$

Since $a^k(\mathbf{m}^k, \mathbf{m}^k) \geq 0$ this implies

$$\int_{\Omega} \phi (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k \, d\mathbf{x} \leq \Delta t f^k(S^k) = \Delta t \int_{\Omega} f^k S^k \, d\mathbf{x}.$$

Estimating the right hand side using Young's inequality, we obtain:

$$\int_{\Omega} f^k S^k \, d\mathbf{x} \leq \int_{\Omega} |f^k| |S^k| \, d\mathbf{x} \leq \int_{\Omega} \frac{1}{3} |f^k|^3 + \frac{2}{3} |S^k|^{3/2} \, d\mathbf{x} = \frac{1}{3} \|f^k\|_{0,3,\Omega}^3 + \frac{2}{3} \|S^k\|_{0,3/2,\Omega}^{3/2}.$$

In a similar manner we can treat the left hand side. Since

$$\left| \int_{\Omega} \phi \rho^{k-1}(S^{k-1}) S^k \, d\mathbf{x} \right| \leq \frac{1}{3} \int_{\Omega} \phi \gamma^{k-1} |S^{k-1}|^{3/2} \, d\mathbf{x} + \frac{2}{3} \int_{\Omega} \phi \gamma^{k-1} |S^k|^{3/2} \, d\mathbf{x},$$

it follows that

$$\int_{\Omega} \phi (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k \, d\mathbf{x} \geq \frac{1}{3} \int_{\Omega} \phi \gamma^k |S^k|^{3/2} \, d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi \gamma^{k-1} |S^{k-1}|^{3/2} \, d\mathbf{x} - \frac{2}{3} \int_{\Omega} \phi (\gamma^{k-1} - \gamma^k) |S^k|^{3/2} \, d\mathbf{x}.$$

Due to the assumptions on γ the integrand in the last term can be bounded by

$$|\phi \gamma^{k-1} - \phi \gamma^k| = \phi |\gamma^{k-1} - \gamma^k| \leq \phi L(\gamma) \Delta t \leq \phi \gamma^k \frac{L(\gamma)}{\underline{\gamma}} \Delta t.$$

Merging all the above estimates together this results in

$$\int_{\Omega} \phi \gamma^k |S^k|^{3/2} \, d\mathbf{x} - \int_{\Omega} \phi \gamma^{k-1} |S^{k-1}|^{3/2} \, d\mathbf{x} \leq \Delta t \|f^k\|_{0,3,\Omega}^3 + 2 \left(\frac{1}{\underline{\phi \gamma}} + \frac{L(\gamma)}{\underline{\gamma}} \right) \Delta t \int_{\Omega} \phi \gamma^k |S^k|^{3/2} \, d\mathbf{x}.$$

If Δt is sufficient small such that $C\Delta t := 2\left(\frac{1}{\underline{\phi}\gamma} + \frac{L(\gamma)}{\underline{\gamma}}\right)\Delta t < 1$, we can conclude that

$$\int_{\Omega} \phi \gamma^k |S^k|^{3/2} d\mathbf{x} \leq \frac{1}{1 - C\Delta t} \left(\int_{\Omega} \phi \gamma^{k-1} |S^{k-1}|^{3/2} d\mathbf{x} + \Delta t \|f^k\|_{0,3,\Omega}^3 \right)$$

for $k = 1, \dots, K$. As $1/(1 - C\Delta t) > 1$, we obtain by induction for all $k = 0, \dots, K$

$$\begin{aligned} \int_{\Omega} \phi \gamma^k |S^k|^{3/2} d\mathbf{x} &\leq (1 - C\Delta t)^{-k} \left(\int_{\Omega} \phi \gamma^0 |S^0|^{3/2} d\mathbf{x} + \sum_{i=1}^k \Delta t \|f^i\|_{0,3,\Omega}^3 \right) \\ &\leq (1 - C\Delta t)^{-K} \left(\int_{\Omega} \phi \gamma^0 |S^0|^{3/2} d\mathbf{x} + TC_f^3 \right), \end{aligned}$$

where C_f is an upper bound for $\|f\|_{0,3,\Omega}$. Note that for $K \rightarrow \infty$ (i.e., $\Delta t = T/K \rightarrow 0$) the expression $(1 - C\Delta t)^{-K} = (1 - CT/K)^{-K}$ tends to e^{CT} . In particular, this expression remains bounded. \square

Lemma 3.2. *For sufficiently small Δt there exists a constant $C_{\mathbf{m}}$, independent of Δt (and of K), such that*

$$\|\mathbf{m}^k\|_{0,3,\Omega} \leq C_{\mathbf{m}} \quad \text{for all } 0 \leq k \leq K. \quad (28)$$

Proof. Choosing $q = S^k - S^{k-1}$ we obtain from the second equation in (25)

$$\int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) (S^k - S^{k-1}) d\mathbf{x} + b(\mathbf{m}^k, S^k - S^{k-1}) = f^k(S^k - S^{k-1}).$$

Since the first term is non-negative, this implies $b(\mathbf{m}^k, S^k - S^{k-1}) \leq f^k(S^k - S^{k-1})$. Furthermore, we choose $\mathbf{v} = \mathbf{m}^k$ in the first equation of (25) belonging to time step k and $k-1$ and subtract the resulting equations. Using the inequality above this yields

$$a^k(\mathbf{m}^k, \mathbf{m}^k) - a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) = b(\mathbf{m}^k, S^k) - b(\mathbf{m}^k, S^{k-1}) \leq f^k(S^k - S^{k-1}).$$

Again, we apply Young's inequality to show

$$\begin{aligned} \left| \int_{\Omega} \alpha^{k-1} (\mathbf{m}^{k-1} \cdot \mathbf{m}^k) d\mathbf{x} \right| &\leq \frac{1}{2} \int_{\Omega} \alpha^k |\mathbf{m}^k|^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} \alpha^{k-1} |\mathbf{m}^{k-1}|^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} (\alpha^{k-1} - \alpha^k) |\mathbf{m}^k|^2 d\mathbf{x}, \\ \left| \int_{\Omega} \beta^{k-1} |\mathbf{m}^{k-1}| (\mathbf{m}^{k-1} \cdot \mathbf{m}^k) d\mathbf{x} \right| &\leq \frac{1}{3} \int_{\Omega} \beta^k |\mathbf{m}^k|^3 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^{k-1} |\mathbf{m}^{k-1}|^3 d\mathbf{x} + \frac{1}{3} \int_{\Omega} (\beta^{k-1} - \beta^k) |\mathbf{m}^k|^3 d\mathbf{x}. \end{aligned}$$

Owing to the definition of a we obtain

$$\begin{aligned} &\frac{1}{2} \int_{\Omega} \alpha^k |\mathbf{m}^k|^2 d\mathbf{x} - \frac{1}{2} \int_{\Omega} \alpha^{k-1} |\mathbf{m}^{k-1}|^2 d\mathbf{x} - \frac{1}{2} \int_{\Omega} (\alpha^{k-1} - \alpha^k) |\mathbf{m}^k|^2 d\mathbf{x} \\ &+ \frac{2}{3} \int_{\Omega} \beta^k |\mathbf{m}^k|^3 d\mathbf{x} - \frac{2}{3} \int_{\Omega} \beta^{k-1} |\mathbf{m}^{k-1}|^3 d\mathbf{x} - \frac{1}{3} \int_{\Omega} (\beta^{k-1} - \beta^k) |\mathbf{m}^k|^3 d\mathbf{x} \\ &\leq a^k(\mathbf{m}^k, \mathbf{m}^k) - a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) \leq f^k(S^k - S^{k-1}). \end{aligned}$$

Summing this relation for $i = 1, \dots, k$ yields

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \alpha^k |\mathbf{m}^k|^2 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^k |\mathbf{m}^k|^3 d\mathbf{x} &\leq \frac{1}{2} \int_{\Omega} \alpha^0 |\mathbf{m}^0|^2 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^0 |\mathbf{m}^0|^3 d\mathbf{x} \\ &+ \sum_{i=1}^k \left[\int_{\Omega} \frac{1}{2} (\alpha^{i-1} - \alpha^i) |\mathbf{m}^i|^2 d\mathbf{x} + \frac{1}{3} (\beta^{i-1} - \beta^i) |\mathbf{m}^i|^3 d\mathbf{x} + f^i(S^i - S^{i-1}) \right]. \end{aligned}$$

This inequality holds for $k = 0, \dots, K$. Due to (26) and the Lipschitz-continuity of f , the last term in this sum is bounded. Indeed, for $k = 1, \dots, K$ it holds:

$$\begin{aligned} \left| \sum_{i=1}^k f^i (S^i - S^{i-1}) \right| &= f^k S^k - f^1 S^0 + \sum_{i=1}^{k-1} (f^i - f^{i+1}) S^i \\ &\leq \|f^k\|_{0,3,\Omega} \|S^k\|_{0,3/2,\Omega} + \|f^1\|_{0,3,\Omega} \|S^0\|_{0,3/2,\Omega} + \sum_{i=1}^{k-1} \|f^i - f^{i+1}\|_{0,3,\Omega} \|S^i\|_{0,3/2,\Omega} \\ &\leq 2C_f C_S + \sum_{i=1}^{k-1} L(f) \Delta t C_S \leq (2C_f + TL(f)) C_S. \end{aligned}$$

Analogously, the estimation of the other terms is based on the Lipschitz-continuity of α and β :

$$\begin{aligned} \int_{\Omega} \frac{1}{2} (\alpha^{i-1} - \alpha^i) |\mathbf{m}^i|^2 + \frac{1}{3} (\beta^{i-1} - \beta^i) |\mathbf{m}^i|^3 d\mathbf{x} &\leq \frac{1}{2} \frac{L(\alpha)}{\underline{\alpha}} \Delta t \int_{\Omega} \alpha^i |\mathbf{m}^i|^2 d\mathbf{x} + \frac{1}{3} \frac{L(\beta)}{\underline{\beta}} \Delta t \int_{\Omega} \beta^i |\mathbf{m}^i|^3 d\mathbf{x} \\ &\leq C(\alpha, \beta) \Delta t \int_{\Omega} \alpha^i |\mathbf{m}^i|^2 + \beta^i |\mathbf{m}^i|^3 d\mathbf{x} = C(\alpha, \beta) \Delta t a^i(\mathbf{m}^i, \mathbf{m}^i), \end{aligned}$$

where $C(\alpha, \beta) := \max \left\{ \frac{1}{2} \frac{L(\alpha)}{\underline{\alpha}}, \frac{1}{3} \frac{L(\beta)}{\underline{\beta}} \right\}$. Summing up, we obtain for $k = 1, \dots, K$

$$\sum_{i=1}^k \int_{\Omega} \frac{1}{2} (\alpha^{i-1} - \alpha^i) |\mathbf{m}^i|^2 + \frac{1}{3} (\beta^{i-1} - \beta^i) |\mathbf{m}^i|^3 d\mathbf{x} \leq C(\alpha, \beta) \sum_{i=1}^k \Delta t a^i(\mathbf{m}^i, \mathbf{m}^i) \leq C(\alpha, \beta) \sum_{i=1}^K \Delta t a^i(\mathbf{m}^i, \mathbf{m}^i).$$

Using (27) finally yields

$$\begin{aligned} \sum_{k=1}^K \Delta t a^k(\mathbf{m}^k, \mathbf{m}^k) &= \sum_{k=1}^K \left[\Delta t f^k(S^k) - \int_{\Omega} \phi (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k d\mathbf{x} \right] \\ &\leq \sum_{k=1}^K \Delta t \|f^k\|_{0,3,\Omega} \|S^k\|_{0,3/2,\Omega} - \sum_{k=1}^K \left[\frac{1}{3} \int_{\Omega} \phi \gamma^k |S^k|^{3/2} d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi \gamma^{k-1} |S^{k-1}|^{3/2} d\mathbf{x} \right] \\ &\quad + \sum_{k=1}^K \frac{2}{3} \int_{\Omega} \phi (\gamma^{k-1} - \gamma^k) |S^k|^{3/2} d\mathbf{x} \\ &\leq \sum_{k=1}^K \Delta t C_f C_S + \frac{1}{3} \int_{\Omega} \phi \gamma^0 |S^0|^{3/2} d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi \gamma^K |S^K|^{3/2} d\mathbf{x} + \frac{2}{3} \bar{\phi} \sum_{k=1}^K \|\gamma^{k-1} - \gamma^k\|_{0,\infty,\Omega} \|S^k\|_{0,3/2,\Omega}^{3/2} \\ &\leq TC_f C_S + \frac{2}{3} \bar{\phi} \bar{\gamma} C_S^{3/2} + \frac{2}{3} \bar{\phi} \sum_{k=1}^K L(\gamma) \Delta t C_S^{3/2} \leq TC_f C_S + \frac{2}{3} \bar{\phi} (\bar{\gamma} + TL(\gamma)) C_S^{3/2}. \end{aligned}$$

Summarizing all the relations above, we obtain the following inequality, which holds for $k = 0, \dots, K$

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \alpha^k |\mathbf{m}^k|^2 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^k |\mathbf{m}^k|^3 d\mathbf{x} &\leq \frac{1}{2} \int_{\Omega} \alpha^0 |\mathbf{m}^0|^2 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^0 |\mathbf{m}^0|^3 d\mathbf{x} \\ &\quad + (2C_f + TL(f) + C(\alpha, \beta) TC_f) C_S + C(\alpha, \beta) \frac{2}{3} \bar{\phi} (\bar{\gamma} + TL(\gamma)) C_S^{3/2}. \end{aligned}$$

Since $\beta^k > \underline{\beta} > 0$, this yields the assertion. \square

Lemma 3.3. *For sufficiently small Δt there exists a constant $C_{S'}$, independent of Δt (and of K), such that*

$$\sum_{k=1}^K \Delta t \int_{\Omega} \left| \frac{S^k - S^{k-1}}{\Delta t} \right|^{3/2} d\mathbf{x} \leq C_{S'} . \quad (29)$$

Proof. In a similar manner as in the proof of Lemma 3.2 we obtain

$$\int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) (S^k - S^{k-1}) d\mathbf{x} + a^k(\mathbf{m}^k, \mathbf{m}^k) - a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) = \int_{\Omega} f^k(S^k - S^{k-1}) d\mathbf{x} .$$

Summing up for $k = 1, \dots, K$ yields

$$\begin{aligned} \sum_{k=1}^K \frac{\phi \gamma}{\Delta t} \int_{\Omega} \left(\frac{S^k}{\sqrt{|S^k|}} - \frac{S^{k-1}}{\sqrt{|S^{k-1}|}} \right) (S^k - S^{k-1}) d\mathbf{x} &\leq \sum_{k=1}^K \left[\int_{\Omega} f^k(S^k - S^{k-1}) d\mathbf{x} - a^k(\mathbf{m}^k, \mathbf{m}^k) + a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) \right] \\ &\leq \left| \sum_{k=1}^K \int_{\Omega} f^k(S^k - S^{k-1}) d\mathbf{x} \right| + \sum_{k=1}^K (a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) - a^k(\mathbf{m}^k, \mathbf{m}^k)) . \end{aligned}$$

As we have seen in the proof of Lemma 3.2, the first term on the right hand side is bounded by $(2C_f + TL(f))C_S$. For the second term the following estimate holds

$$\begin{aligned} &\sum_{k=1}^K (a^{k-1}(\mathbf{m}^{k-1}, \mathbf{m}^k) - a^k(\mathbf{m}^k, \mathbf{m}^k)) \\ &\leq \sum_{k=1}^K \left(\frac{1}{2} \int_{\Omega} \alpha^{k-1} |\mathbf{m}^{k-1}|^2 d\mathbf{x} - \frac{1}{2} \int_{\Omega} \alpha^k |\mathbf{m}^k|^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} (\alpha^{k-1} - \alpha^k) |\mathbf{m}^k|^2 d\mathbf{x} \right. \\ &\quad \left. + \frac{2}{3} \int_{\Omega} \beta^{k-1} |\mathbf{m}^{k-1}|^3 d\mathbf{x} - \frac{2}{3} \int_{\Omega} \beta^k |\mathbf{m}^k|^3 d\mathbf{x} + \frac{1}{3} \int_{\Omega} (\beta^{k-1} - \beta^k) |\mathbf{m}^k|^3 d\mathbf{x} \right) \\ &= \frac{1}{2} \int_{\Omega} \alpha^0 |\mathbf{m}^0|^2 d\mathbf{x} - \frac{1}{2} \int_{\Omega} \alpha^K |\mathbf{m}^K|^2 d\mathbf{x} + \frac{2}{3} \int_{\Omega} \beta^0 |\mathbf{m}^0|^3 d\mathbf{x} - \frac{2}{3} \int_{\Omega} \beta^K |\mathbf{m}^K|^3 d\mathbf{x} \\ &\quad + \sum_{k=1}^K \left(\frac{1}{2} \int_{\Omega} (\alpha^{k-1} - \alpha^k) |\mathbf{m}^k|^2 d\mathbf{x} + \frac{1}{3} \int_{\Omega} (\beta^{k-1} - \beta^k) |\mathbf{m}^k|^3 d\mathbf{x} \right) \\ &\leq \frac{1}{2} C(\bar{\alpha}) \|\mathbf{m}^0\|_{0,3,\Omega}^2 + \frac{2}{3} C(\bar{\beta}) \|\mathbf{m}^0\|_{0,3,\Omega}^3 + C(\alpha, \beta) \left(TC_f C_S + \frac{2}{3} \bar{\phi} (\bar{\gamma} + TL(\gamma)) C_S^{3/2} \right) . \end{aligned}$$

Using (15) we therefore showed that there exists a constant $C > 0$, independent of Δt , such that

$$\sum_{k=1}^K \Delta t \int_{\Omega} \frac{1}{\sqrt{|S^k|} + \sqrt{|S^{k-1}|}} \left(\frac{S^k - S^{k-1}}{\Delta t} \right)^2 d\mathbf{x} \leq \sum_{k=1}^K \frac{1}{\Delta t} \int_{\Omega} \left(\frac{S^k}{\sqrt{|S^k|}} - \frac{S^{k-1}}{\sqrt{|S^{k-1}|}} \right) (S^k - S^{k-1}) d\mathbf{x} \leq C .$$

Applying Hölder's inequality finally yields the assertion:

$$\begin{aligned}
\sum_{k=1}^K \Delta t \int_{\Omega} \left| \frac{S^k - S^{k-1}}{\Delta t} \right|^{3/2} d\mathbf{x} &= \sum_{k=1}^K \Delta t \int_{\Omega} \left(\sqrt{|S^k|} + \sqrt{|S^{k-1}|} \right)^{3/4} \left(\frac{1}{\sqrt{|S^k|} + \sqrt{|S^{k-1}|}} \left(\frac{S^k - S^{k-1}}{\Delta t} \right)^2 \right)^{3/4} d\mathbf{x} \\
&\leq \sum_{k=1}^K \left[\left(\Delta t \int_{\Omega} \left(\sqrt{|S^k|} + \sqrt{|S^{k-1}|} \right)^3 d\mathbf{x} \right)^{1/4} \left(\Delta t \int_{\Omega} \frac{1}{\sqrt{|S^k|} + \sqrt{|S^{k-1}|}} \left(\frac{S^k - S^{k-1}}{\Delta t} \right)^2 d\mathbf{x} \right)^{3/4} \right] \\
&\leq \left(\sum_{k=1}^K \Delta t \int_{\Omega} \left(\sqrt{|S^k|} + \sqrt{|S^{k-1}|} \right)^3 d\mathbf{x} \right)^{1/4} \left(\sum_{k=1}^K \Delta t \int_{\Omega} \frac{1}{\sqrt{|S^k|} + \sqrt{|S^{k-1}|}} \left(\frac{S^k - S^{k-1}}{\Delta t} \right)^2 d\mathbf{x} \right)^{3/4} \\
&\leq T^{1/4} 2^{3/4} C_S^{3/8} C^{3/4} =: C_{S'}.
\end{aligned}$$

□

Next, we show that the mixed formulation (25) is equivalent to a variational formulation of the time-discretized parabolic equation (1). To this end, we recall the nonlinear mapping F of (3). For fixed time $t = t_k$, we define the nonlinear mapping $F^k : (L^{3/2}(\Omega))^n \rightarrow (L^3(\Omega))^n$ and its inverse G^k defined by $G^k(\mathbf{v}) = (\alpha^k + \beta^k |\mathbf{v}|) \mathbf{v}$. Note that $\int_{\Omega} G^k(\mathbf{u}) \cdot \mathbf{v} d\mathbf{x} = a^k(\mathbf{u}, \mathbf{v})$ for $\mathbf{u}, \mathbf{v} \in (L^3(\Omega))^n$.

Proposition 3.4. (a) *If $S^k \in W^{1,3/2}(\Omega)$ is a solution of the variational formulation: Find $S^k \in W_0^{1,3/2}(\Omega)$ such that*

$$\int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q d\mathbf{x} + \int_{\Omega} F^k(\nabla S^k) \cdot \nabla q d\mathbf{x} = f^k(q) \quad \text{for all } q \in W_0^{1,3/2}(\Omega), \quad (30)$$

then $(F^k(\nabla S^k), S^k)$ is a solution of the mixed formulation (25). In particular, $F^k(\nabla S^k) \in W^3(\text{div}; \Omega)$.

(b) *If $(\mathbf{m}^k, S^k) \in W^3(\text{div}; \Omega) \times L^{3/2}(\Omega)$ is a solution of the mixed formulation (25), then S^k is a solution of the variational formulation (30). In particular, $S^k \in W_0^{1,3/2}(\Omega)$.*

Proof. Ad a) Let S^k be a solution of (30). We define $\mathbf{m}^k := F(\nabla S^k)$. Then Green's formula yields

$$\int_{\Omega} G^k(\mathbf{m}^k) \cdot \mathbf{v} d\mathbf{x} = \int_{\Omega} G^k(F^k(\nabla S^k)) \cdot \mathbf{v} d\mathbf{x} = \int_{\Omega} \nabla S^k \cdot \mathbf{v} d\mathbf{x} = - \int_{\Omega} \text{div}(\mathbf{v}) S^k d\mathbf{x} \quad \text{for all } \mathbf{v} \in W^3(\text{div}; \Omega).$$

This is the first equation in (25). To derive the second equation in (25), we consider (30) for $q \in \mathcal{D}(\Omega) \subset W_0^{1,3/2}(\Omega)$, and apply Green's formula again:

$$\int_{\Omega} (\rho^k(S^k) - \rho^{k-1}(S^{k-1}) - f^k) q d\mathbf{x} = - \int_{\Omega} F^k(\nabla S^k) \cdot \nabla q d\mathbf{x} = - \int_{\Omega} \mathbf{m}^k \cdot \nabla q d\mathbf{x}.$$

Thus the difference $\rho^k(S^k) - \rho^{k-1}(S^{k-1}) - f \in L^3(\Omega)$ is the generalized divergence of \mathbf{m}^k ; consequently $\mathbf{m}^k \in W^3(\text{div}; \Omega)$. Because $\mathcal{D}(\Omega)$ is densely embedded into $L^{3/2}(\Omega)$, the second equation in (25) follows.

Ad b) Now, let (\mathbf{m}^k, S^k) be a solution of (25). Green's formula then implies

$$\int_{\Omega} G^k(\mathbf{m}^k) \cdot \mathbf{v} d\mathbf{x} = - \int_{\Omega} \text{div}(\mathbf{v}) S^k d\mathbf{x} \quad \text{for all } \mathbf{v} \in (\mathcal{D}(\Omega))^n.$$

Thus in the sense of distributions it holds $\nabla S^k = G^k(\mathbf{m}^k) \in (L^{3/2}(\Omega))^n$. Consequently, $S^k \in W^{1,3/2}(\Omega)$ and $\mathbf{m}^k = F^k(\nabla S^k)$. To prove that S^k fulfills (30), we consider $q \in W_0^{1,3/2}(\Omega) \subset L^{3/2}(\Omega)$ in the first equation of

(25). Another application of Green's formula yields

$$\begin{aligned} f^k(q) &= \int_{\Omega} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q \, d\mathbf{x} - \int_{\Omega} \operatorname{div}(F^k(\nabla S^k)) q \, d\mathbf{x} \\ &= \int_{\Omega} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q \, d\mathbf{x} + \int_{\Omega} F^k(\nabla S^k) \cdot \nabla q \, d\mathbf{x}. \end{aligned}$$

Finally, we consider again the first equation of (25) for $\mathbf{v} \in (\mathcal{D}(\bar{\Omega}))^n$. Applying Green's formula we obtain

$$0 = \int_{\Omega} \nabla S^k \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Omega} \operatorname{div}(\mathbf{v}) S^k \, d\mathbf{x} = \int_{\partial\Omega} \gamma_0 S^k (\mathbf{v} \cdot \mathbf{n}) \, d\sigma.$$

Consequently, $\gamma_0 S^k = 0$ in $W^{1,3/2}(\partial\Omega)$, i.e. $S^k \in W_0^{1,3/2}(\Omega)$. \square

Using this equivalence, we obtain a bound for S^k in the norm of $W^{1,3/2}(\Omega)$.

Lemma 3.5. *For sufficiently small Δt there exist constants \bar{C}_S , $C_{\rho'}$ and $C_{\bar{\mathbf{m}}}$, all independent of Δt (and of K), such that*

$$\|S^k\|_{1,3/2,\Omega} \leq \bar{C}_S \quad \text{for all } 0 \leq k \leq K, \quad (31)$$

$$\left\| \frac{\rho^k(S^k) - \rho^{k-1}(S^{k-1})}{\Delta t} \right\|_{-1,3,\Omega} \leq C_{\rho'} \quad \text{for all } 1 \leq k \leq K, \quad (32)$$

$$\|\operatorname{div}(\mathbf{m}^k)\|_{-1,3,\Omega} \leq C_{\bar{\mathbf{m}}} \quad \text{for all } 1 \leq k \leq K. \quad (33)$$

Proof. Proposition 3.4 gives $\nabla S^k = -G^k(\mathbf{m}^k)$. Therefore (28) implies

$$\|\nabla S^k\|_{0,3/2,\Omega} = \|G^k(\mathbf{m}^k)\|_{0,3/2,\Omega} \leq C(\bar{\alpha})C_{\mathbf{m}} + C(\bar{\beta})C_{\mathbf{m}}^2 =: C_G.$$

Together with (26) we obtain (31).

Also, this equivalence yields (32), because by means of (30) we have for all $q \in W_0^{1,3/2}(\Omega)$

$$\begin{aligned} \left| \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q \, d\mathbf{x} \right| &= \left| f^k(q) - \int_{\Omega} F^k(\nabla S^k) \cdot \nabla q \, d\mathbf{x} \right| = \left| f^k(q) + \int_{\Omega} \mathbf{m}^k \cdot \nabla q \, d\mathbf{x} \right| \\ &\leq \|f^k\|_{0,3,\Omega} \|q\|_{0,3/2,\Omega} + \|\mathbf{m}^k\|_{0,3,\Omega} \|\nabla q\|_{0,3/2,\Omega} \\ &\leq (\|f^k\|_{0,3,\Omega} + \|\mathbf{m}^k\|_{0,3,\Omega}) \|q\|_{1,3/2,\Omega}. \end{aligned}$$

Finally, we obtain (33), since the first equation of (25) yields

$$\left| \int_{\Omega} \operatorname{div}(\mathbf{m}^k) q \, d\mathbf{x} \right| = \left| f^k(q) - \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) q \, d\mathbf{x} \right| \leq (\|f^k\|_{0,3,\Omega} + \bar{\phi} C_{\rho'}) \|q\|_{1,3/2,\Omega}.$$

for all $q \in W_0^{1,3/2}(\Omega)$ \square

3.2. Solvability of the continuous problem

Due to the existence of unique solutions to the semi-discrete mixed formulation (25) we obtain for every $K \in \mathbb{N}$ a $K+1$ -tuple of solutions $((\mathbf{m}_{\Delta t}^k, S_{\Delta t}^k))_{k=0,\dots,K} \in (W^3(\operatorname{div}; \Omega) \times L^{3/2}(\Omega))^{K+1}$. Recall that $\Delta t = T/K$. We denote these $K+1$ -tuples with $\mathbf{m}_{\Delta t} := (\mathbf{m}_{\Delta t}^k)_{k=0,\dots,K} \in (W^3(\operatorname{div}; \Omega))^{K+1}$ and $S_{\Delta t} := (S_{\Delta t}^k)_{k=0,\dots,K} \in$

$(L^{3/2}(\Omega))^{K+1}$. We define step functions, e.g. $\Pi_{\Delta t} S_{\Delta t} \in L^\infty(0, T; W_0^{1,3/2}(\Omega))$, which are piecewise constant in time, by

$$(\Pi_{\Delta t} S_{\Delta t})(t) := \begin{cases} S_{\Delta t}^0, & \text{if } t = 0, \\ S_{\Delta t}^k, & \text{if } (k-1)\Delta t < t \leq k\Delta t, \quad k = 1, \dots, K, \end{cases}$$

and piecewise linear (in time) functions $\Lambda_{\Delta t} S_{\Delta t} \in C([0, T]; W_0^{1,3/2}(\Omega))$ fulfilling

$$(\Lambda_{\Delta t} S_{\Delta t})(t^k) = S_{\Delta t}^k \quad \text{for } k = 0, \dots, K.$$

The time derivative of $\Lambda_{\Delta t} S_{\Delta t}$ is a piecewise constant step function with values

$$\Lambda'_{\Delta t} S_{\Delta t}(t) := \frac{\partial}{\partial t} (\Lambda_{\Delta t} S_{\Delta t})(t) = \frac{S_{\Delta t}^k - S_{\Delta t}^{k-1}}{\Delta t}, \quad \text{if } (k-1)\Delta t < t < k\Delta t, \quad k = 1, \dots, K.$$

In addition, we use piecewise constant approximations $\gamma_{\Delta t}$ and $f_{\Delta t}$ of the coefficient functions γ and f , and piecewise constant operators $\rho_{\Delta t}$, $F_{\Delta t}$ and $G_{\Delta t}$. Owing to the lemmas above the following bounds hold for sufficiently small time step sizes Δt :

$$\begin{aligned} \|\Pi_{\Delta t} S_{\Delta t}\|_{L^\infty(0, T; W_0^{1,3/2}(\Omega))} &\leq \overline{C}_S, \\ \|\Lambda'_{\Delta t} S_{\Delta t}\|_{L^{3/2}(0, T; L^{3/2}(\Omega))} &\leq C_{S'}, \\ \|\Pi_{\Delta t} \rho_{\Delta t}(S_{\Delta t})\|_{L^\infty(0, T; L^3(\Omega))} &\leq \overline{\gamma} \sqrt{C_S}, \\ \|\Lambda'_{\Delta t} \rho_{\Delta t}(S_{\Delta t})\|_{L^\infty(0, T; W^{-1,3}(\Omega))} &\leq C_{\rho'}, \\ \|\Pi_{\Delta t} \mathbf{m}_{\Delta t}\|_{L^\infty(0, T; (L^3(\Omega))^n)} &\leq C_{\mathbf{m}}, \\ \|\Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t})\|_{L^\infty(0, T; W^{-1,3}(\Omega))} &\leq C_{\overline{\mathbf{m}}}, \\ \|\Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t})\|_{L^\infty(0, T; (L^{3/2}(\Omega))^n)} &\leq C_G, \\ \|\rho^K(S^K)\|_{0,3,\Omega} &\leq \overline{\gamma} \sqrt{C_S}. \end{aligned}$$

The third (and the last) inequality follow from

$$\int_{\Omega} |\rho^k(S^k)|^3 d\mathbf{x} = \int_{\Omega} \left| \gamma^k \frac{S^k}{\sqrt{|S^k|}} \right|^3 d\mathbf{x} \leq \overline{\gamma}^3 \int_{\Omega} |S^k|^{3/2} d\mathbf{x} \leq \overline{\gamma}^3 C_S^{3/2}.$$

Thus there exist subsequences, again indexed by Δt , that converge in the corresponding weak*-topology; in detail

$$\begin{aligned} \Pi_{\Delta t} S_{\Delta t} &\overset{*}{\rightharpoonup} S && \text{in } L^\infty(0, T; W_0^{1,3/2}(\Omega)), \\ \Lambda'_{\Delta t} S_{\Delta t} &\rightharpoonup S' && \text{in } L^{3/2}(0, T; L^{3/2}(\Omega)), \\ \Pi_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) &\overset{*}{\rightharpoonup} R && \text{in } L^\infty(0, T; L^3(\Omega)), \\ \Lambda'_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) &\overset{*}{\rightharpoonup} R' && \text{in } L^\infty(0, T; W^{-1,3}(\Omega)), \\ \Pi_{\Delta t} \mathbf{m}_{\Delta t} &\overset{*}{\rightharpoonup} \mathbf{m} && \text{in } L^\infty(0, T; (L^3(\Omega))^n), \\ \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}) &\overset{*}{\rightharpoonup} \overline{\mathbf{m}} && \text{in } L^\infty(0, T; W^{-1,3}(\Omega)), \\ \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) &\overset{*}{\rightharpoonup} \mathbf{g} && \text{in } L^\infty(0, T; (L^{3/2}(\Omega))^n), \\ \rho^K(S^K) &\rightharpoonup R_T && \text{in } L^3(\Omega). \end{aligned} \tag{34}$$

Proposition 3.6. *The limits S of $\Pi_{\Delta t} S_{\Delta t}$ and R of $\Pi_{\Delta t} \rho_{\Delta t}(S_{\Delta t})$ from (34) satisfy $\rho(S) = R$ almost everywhere in $(0, T) \times \Omega$.*

Proof. As well as $\Pi_{\Delta t} S_{\Delta t}$, also $\Lambda_{\Delta t} S_{\Delta t}$ is bounded in $L^\infty(0, T; W_0^{1,3/2}(\Omega))$. In particular, $\Lambda_{\Delta t} S_{\Delta t}$ and its partial derivatives $(\partial/\partial x_i) \Lambda_{\Delta t} S_{\Delta t}$ are bounded in $L^{3/2}(0, T; L^{3/2}(\Omega))$. Owing to (29) $(\partial/\partial t) \Lambda_{\Delta t} S_{\Delta t} = \Lambda'_{\Delta t} S_{\Delta t}$ is bounded, too, such that $\Lambda_{\Delta t} S_{\Delta t}$ is bounded in $W^{1,3/2}((0, T) \times \Omega)$. The Rellich–Kondrachov-Theorem yields that $W^{1,3/2}$ is embedded compactly in $L^{3/2}$. Thus there exists a subsequence (again denoted by $\Lambda_{\Delta t}$), which converges (strongly) in $L^{3/2}((0, T) \times \Omega)$ to S . Choosing a further subsequence, we obtain that $\Lambda_{\Delta t} S_{\Delta t}$ converges almost everywhere in $(0, T) \times \Omega$ to S . Applying the mapping $\rho_{\Delta t}$ yields $\lim_{\Delta t \rightarrow 0} \Lambda_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) = \rho(S)$ a.e. in $(0, T) \times \Omega$. Since $\Lambda_{\Delta t} \rho_{\Delta t}(S_{\Delta t})$ is bounded in $L^3((0, T) \times \Omega)$, we can conclude that $\Lambda_{\Delta t} \rho_{\Delta t}(S_{\Delta t})$ weakly converges to $\rho(S)$ in $L^3((0, T) \times \Omega)$, i.e. for all $q \in L^{3/2}((0, T) \times \Omega)$ it holds

$$\lim_{\Delta t \rightarrow 0} \int_0^T \int_\Omega \Lambda_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) q \, d\mathbf{x} \, dt = \int_0^T \int_\Omega \rho(S) q \, d\mathbf{x} \, dt.$$

On the other hand

$$\lim_{\Delta t \rightarrow 0} \int_0^T \int_\Omega \Lambda_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) q \, d\mathbf{x} \, dt = \lim_{\Delta t \rightarrow 0} \int_0^T \int_\Omega \Pi_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) q \, d\mathbf{x} \, dt = \int_0^T \int_\Omega R q \, d\mathbf{x} \, dt.$$

Since the limit of a convergent sequence is unique, the assertion follows. \square

For the remainder of this section, we denote by $\langle \cdot, \cdot \rangle$ the dual pairing between $W^{-1,3}(\Omega)$ and $W_0^{1,3/2}(\Omega)$.

Proposition 3.7. a) *The identity $S' = (\partial/\partial t)S$ holds in the sense of distributions from $(0, T)$ to $L^{3/2}(\Omega)$, i.e., for all $\varphi \in \mathcal{D}((0, T))$ it holds:*

$$\int_0^T S'(t) \varphi(t) \, dt = - \int_0^T S(t) \varphi'(t) \, dt \quad \text{in } L^{3/2}(\Omega).$$

b) *The identity $R' = (\partial/\partial t)R$ holds in the sense of distributions from $(0, T)$ to $W^{-1,3}(\Omega)$, i.e., for all $\varphi \in \mathcal{D}((0, T))$ it holds:*

$$\int_0^T R'(t) \varphi(t) \, dt = - \int_0^T R(t) \varphi'(t) \, dt \quad \text{in } W^{-1,3}(\Omega).$$

c) *The identity $\overline{\mathbf{m}} = \operatorname{div}(\mathbf{m})$ in the sense of distributions on Ω holds almost everywhere in $(0, T)$, i.e., for all $\Psi \in \mathcal{D}(\Omega)$ it holds:*

$$\langle \overline{\mathbf{m}}, \Psi \rangle = - \int_\Omega \mathbf{m} \cdot \nabla \Psi \, d\mathbf{x} \quad \text{a.e. in } (0, T).$$

d) *The identity $\mathbf{g} = -\nabla S$ holds in $L^\infty(0, T; (L^{3/2}(\Omega))^n)$, i.e., for all $\mathbf{v} \in L^1(0, T; (L^3(\Omega))^n)$ it holds:*

$$\int_0^T \int_\Omega \mathbf{g} \cdot \mathbf{v} \, d\mathbf{x} \, dt = - \int_0^T \int_\Omega \nabla S \cdot \mathbf{v} \, d\mathbf{x} \, dt.$$

Proof. Ad a) From the second equation in (34) we can conclude that for all $\varphi \in \mathcal{D}((0, T))$

$$\lim_{\Delta t \rightarrow 0} \int_0^T \Lambda'_{\Delta t} S_{\Delta t}(t) \varphi(t) \, dt = \int_0^T S'(t) \varphi(t) \, dt \quad \text{in } L^{3/2}(\Omega).$$

On the other hand, partial integration yields

$$\begin{aligned}
\int_0^T \Lambda'_{\Delta t} S_{\Delta t}(t) \varphi(t) dt &= \sum_{k=1}^K \int_{(k-1)\Delta t}^{k\Delta t} \Lambda'_{\Delta t} S_{\Delta t}(t) \varphi(t) dt \\
&= \sum_{k=1}^K \left(\Lambda_{\Delta t} S_{\Delta t}(k\Delta t) \varphi(k\Delta t) - \Lambda_{\Delta t} S_{\Delta t}((k-1)\Delta t) \varphi((k-1)\Delta t) \right) - \sum_{k=1}^K \int_{(k-1)\Delta t}^{k\Delta t} \Lambda_{\Delta t} S_{\Delta t}(t) \varphi'(t) dt \\
&= - \int_0^T \Lambda_{\Delta t} S_{\Delta t}(t) \varphi'(t) dt,
\end{aligned}$$

such that

$$\begin{aligned}
\int_0^T S'(t) \varphi(t) dt &= \lim_{\Delta t \rightarrow 0} \int_0^T \Lambda'_{\Delta t} S_{\Delta t}(t) \varphi(t) dt = \lim_{\Delta t \rightarrow 0} - \int_0^T \Lambda_{\Delta t} S_{\Delta t}(t) \varphi'(t) dt \\
&= \lim_{\Delta t \rightarrow 0} - \int_0^T \Pi_{\Delta t} S_{\Delta t}(t) \varphi'(t) dt = - \int_0^T S(t) \varphi'(t) dt.
\end{aligned}$$

The identity in b) follows in a similar manner as the identity in a).

Ad c) Let $\Psi \in \mathcal{D}(\Omega)$ and $\varphi \in \mathcal{D}((0, T))$ be arbitrarily chosen. Then

$$\begin{aligned}
\lim_{\Delta t \rightarrow 0} \int_0^T \langle \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}), \Psi \rangle \varphi(t) dt &= \lim_{\Delta t \rightarrow 0} \int_0^T \langle \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}), \varphi(t) \Psi \rangle dt \\
&= \int_0^T \langle \overline{\mathbf{m}}(t), \varphi(t) \Psi \rangle dt = \int_0^T \langle \overline{\mathbf{m}}(t), \Psi \rangle \varphi(t) dt,
\end{aligned}$$

because $\overline{\mathbf{m}}$ is the limit of $(\Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}))_{\Delta t}$ in $L^\infty(0, T; W^{-1,3}(\Omega))$. On the other hand

$$\begin{aligned}
\int_0^T \langle \overline{\mathbf{m}}(t), \Psi \rangle \varphi(t) dt &= \lim_{\Delta t \rightarrow 0} \int_0^T \langle \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}), \Psi \rangle \varphi(t) dt = \lim_{\Delta t \rightarrow 0} - \int_0^T \int_\Omega \Pi_{\Delta t} \mathbf{m}_{\Delta t} \cdot \nabla \Psi d\mathbf{x} \varphi(t) dt \\
&= \lim_{\Delta t \rightarrow 0} - \int_0^T \int_\Omega \Pi_{\Delta t} \mathbf{m}_{\Delta t} \cdot \nabla \Psi \varphi(t) d\mathbf{x} dt = - \int_0^T \int_\Omega \mathbf{m} \cdot \nabla \Psi \varphi(t) d\mathbf{x} dt \\
&= - \int_0^T \int_\Omega \mathbf{m} \cdot \nabla \Psi d\mathbf{x} \varphi(t) dt.
\end{aligned}$$

Since φ is arbitrarily chosen, the assertion follows.

Ad d) We have seen in the proof of Proposition 3.4 that

$$\int_\Omega G^k(\mathbf{m}^k) \cdot \mathbf{v} d\mathbf{x} = - \int_\Omega \nabla S^k \cdot \mathbf{v} d\mathbf{x} \quad \text{for all } \mathbf{v} \in (L^3(\Omega))^n.$$

Consequently, for $\mathbf{v} \in L^1(0, T; (L^3(\Omega))^n)$ it holds

$$\begin{aligned}
\int_0^T \int_\Omega \mathbf{g} \cdot \mathbf{v} d\mathbf{x} dt &= \lim_{\Delta t \rightarrow 0} \int_0^T \int_\Omega \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) \cdot \mathbf{v} d\mathbf{x} dt \\
&= \lim_{\Delta t \rightarrow 0} - \int_0^T \int_\Omega \nabla(\Pi_{\Delta t} S_{\Delta t}) \cdot \mathbf{v} d\mathbf{x} dt = - \int_0^T \int_\Omega \nabla S \cdot \mathbf{v} d\mathbf{x} dt.
\end{aligned}$$

□

The first two statements of Proposition 3.7 imply that after possible modification on a set of measure zero in $[0, T]$ we have that $S \in C([0, T]; L^{3/2}(\Omega))$ and $R \in C([0, T]; W^{-1,3}(\Omega))$. Thus for every $t \in [0, T]$ the value $S(t) \in L^{3/2}(\Omega)$ is well defined.

Proposition 3.8. *The following identity holds in $L^\infty(0, T; W^{-1,3}(\Omega))$:*

$$\phi \frac{\partial \rho(S)}{\partial t} + \operatorname{div}(\mathbf{m}) = f.$$

Furthermore $\rho(S(0)) = \rho^0(S^0)$ and $\rho(S(T)) = R_T$.

Proof. For $\varphi \in \mathcal{D}(\overline{(0, T)})$ we define a step function $\varphi_{\Delta t}$ by

$$\varphi_{\Delta t}(t) := \begin{cases} \varphi((k-1)\Delta t), & \text{if } (k-1)\Delta t \leq t < k\Delta t, \ k = 1, \dots, K, \\ \varphi(T), & \text{if } t = T. \end{cases}$$

Using the test function $q = \Psi \in \mathcal{D}(\Omega)$ in (25), multiplying by $\Delta t \varphi((k-1)\Delta t)$ and summing up for $k = 1, \dots, K$, we obtain

$$\begin{aligned} \sum_{k=1}^K \Delta t \int_{\Omega} \phi \frac{\rho^k(S^k) - \rho^{k-1}(S^{k-1})}{\Delta t} \Psi \, d\mathbf{x} \varphi((k-1)\Delta t) \\ + \sum_{k=1}^K \Delta t \int_{\Omega} \operatorname{div}(\mathbf{m}^k) \Psi \, d\mathbf{x} \varphi((k-1)\Delta t) = \sum_{k=1}^K \Delta t \int_{\Omega} f^k \Psi \, d\mathbf{x} \varphi((k-1)\Delta t) \end{aligned} \quad (35)$$

Employing the piecewise constant functions $\Pi_{\Delta t}$ and $\Lambda'_{\Delta t}$ this reads

$$\begin{aligned} \sum_{k=1}^K \int_{(k-1)\Delta t}^{k\Delta t} \int_{\Omega} \phi \Lambda'_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) \Psi \, d\mathbf{x} \varphi_{\Delta t} \, dt \\ + \sum_{k=1}^K \int_{(k-1)\Delta t}^{k\Delta t} \int_{\Omega} \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}) \Psi \, d\mathbf{x} \varphi_{\Delta t} \, dt = \sum_{k=1}^K \int_{(k-1)\Delta t}^{k\Delta t} \int_{\Omega} \Pi_{\Delta t} f_{\Delta t} \Psi \, d\mathbf{x} \varphi_{\Delta t} \, dt, \end{aligned}$$

and after joining the integrals over t

$$\int_0^T \int_{\Omega} \phi \Lambda'_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) \Psi \varphi_{\Delta t} \, d\mathbf{x} \, dt + \int_0^T \int_{\Omega} \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}) \Psi \varphi_{\Delta t} \, d\mathbf{x} \, dt = \int_0^T \int_{\Omega} \Pi_{\Delta t} f_{\Delta t} \Psi \varphi_{\Delta t} \, d\mathbf{x} \, dt.$$

Since $(\Psi \varphi_{\Delta t})_{\Delta t}$ strongly converges to $\Psi \varphi$ in $L^1(0, T; W_0^{1,3/2}(\Omega))$, we can pass to the limit $\Delta t \rightarrow 0$:

$$\int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, \Psi \varphi \right\rangle dt + \int_0^T \langle (\mathbf{m}), \Psi \varphi \rangle dt = \int_0^T \int_{\Omega} f \Psi \varphi \, d\mathbf{x} \, dt. \quad (36)$$

But the set $\left\{ \Psi(\mathbf{x})\varphi(t) \mid \Psi \in \mathcal{D}(\Omega), \varphi \in \mathcal{D}(\overline{(0, T)}) \right\}$ is a dense subset of $L^1(0, T; W_0^{1,3/2}(\Omega))$. Therefore the first identity in Proposition (3.8) is established.

To prove the remaining two identities we first conclude from (35) that

$$\begin{aligned} \sum_{k=1}^K \left(\int_{\Omega} \phi \rho^k(S^k) \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t) - \int_{\Omega} \phi \rho^{k-1}(S^{k-1}) \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t) \right) \\ + \sum_{k=1}^K \Delta t \int_{\Omega} \operatorname{div}(\mathbf{m}^k) \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t) = \sum_{k=1}^K \Delta t \int_{\Omega} f^k \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t). \end{aligned}$$

Rearranging the terms in the first line yields

$$\begin{aligned} \sum_{k=1}^K \left(\int_{\Omega} \phi \rho^k(S^k) \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t) - \int_{\Omega} \phi \rho^{k-1}(S^{k-1}) \Psi \, d\mathbf{x} \, \varphi((k-1)\Delta t) \right) \\ = - \sum_{k=1}^K \int_{\Omega} \phi \rho^k(S^k) \Psi \, d\mathbf{x} \left(\varphi(k\Delta t) - \varphi((k-1)\Delta t) \right) \\ + \int_{\Omega} \phi \rho^K(S^K) \Psi \, d\mathbf{x} \, \varphi(K\Delta t) - \int_{\Omega} \phi \rho^0(S^0) \Psi \, d\mathbf{x} \, \varphi(0). \end{aligned}$$

Like above this leads to

$$\begin{aligned} - \int_0^T \int_{\Omega} \phi \Pi_{\Delta t} \rho_{\Delta t}(S_{\Delta t}) \Psi \, d\mathbf{x} \, \frac{\varphi(k\Delta t) - \varphi((k-1)\Delta t)}{\Delta t} \, dt + \int_0^T \int_{\Omega} \Pi_{\Delta t} \operatorname{div}(\mathbf{m}_{\Delta t}) \Psi \, d\mathbf{x} \, \varphi_{\Delta t} \, dt \\ = \int_0^T \int_{\Omega} \Pi_{\Delta t} f_{\Delta t} \Psi \, d\mathbf{x} \, \varphi_{\Delta t} \, dt + \int_{\Omega} \phi \rho^0(S^0) \Psi \, d\mathbf{x} \, \varphi(0) - \int_{\Omega} \phi \rho^K(S^K) \Psi \, d\mathbf{x} \, \varphi(T). \end{aligned}$$

Passing to the limit $\Delta t \rightarrow 0$ we obtain

$$- \int_0^T \int_{\Omega} \phi \rho(S) \Psi \, d\mathbf{x} \, \frac{\partial \varphi}{\partial t} \, dt + \int_0^T \langle (\mathbf{m}), \Psi \rangle \varphi \, dt = \int_0^T \int_{\Omega} f \Psi \, d\mathbf{x} \, \varphi \, dt + \int_{\Omega} \phi \rho^0(S^0) \Psi \, d\mathbf{x} \, \varphi(0) - \int_{\Omega} \phi R_T \Psi \, d\mathbf{x} \, \varphi(T).$$

In the other hand, partial integration of (36) yields

$$- \int_0^T \int_{\Omega} \phi \rho(S) \Psi \, d\mathbf{x} \, \frac{\partial \varphi}{\partial t} \, dt + \int_0^T \langle \operatorname{div}(\mathbf{m}), \Psi \rangle \varphi \, dt = \int_0^T \int_{\Omega} f \Psi \, d\mathbf{x} \, \varphi \, dt + \int_{\Omega} \phi \rho(S(0)) \Psi \, d\mathbf{x} \, \varphi(0) - \int_{\Omega} \phi \rho(S(T)) \Psi \, d\mathbf{x} \, \varphi(T).$$

Subtracting the last two equations we can conclude

$$\left(\int_{\Omega} \phi \rho^0(S^0) \Psi \, d\mathbf{x} - \int_{\Omega} \phi \rho(S(0)) \Psi \, d\mathbf{x} \right) \varphi(0) - \left(\int_{\Omega} \phi R_T \Psi \, d\mathbf{x} - \int_{\Omega} \phi \rho(S(T)) \Psi \, d\mathbf{x} \right) \varphi(T) = 0.$$

Since $\varphi(0)$ and $\varphi(T)$ are arbitrary, this implies

$$\int_{\Omega} \phi \rho^0(S^0) \Psi \, d\mathbf{x} = \int_{\Omega} \phi \rho(S(0)) \Psi \, d\mathbf{x} \quad \text{and} \quad \int_{\Omega} \phi R_T \Psi \, d\mathbf{x} = \int_{\Omega} \phi \rho(S(T)) \Psi \, d\mathbf{x}$$

and finally $\rho(S(0)) = \rho^0(S^0)$ and $\rho(S(T)) = R_T$. \square

Only the identity $\mathbf{g} = G(\mathbf{m})$ is missing yet. To show this, we need an auxiliary result, a generalization of Lemma 1.2 from [7].

Lemma 3.9. *The limit S of $(\Pi_{\Delta t} S_{\Delta t})_{\Delta t}$ satisfies:*

$$\int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt = \frac{1}{3} \left(\int_{\Omega} \phi |\rho(S(T))| |S(T)| d\mathbf{x} - \int_{\Omega} \phi |\rho(S(0))| |S(0)| d\mathbf{x} \right) + \frac{2}{3} \int_0^T \int_{\Omega} \phi \frac{\partial \gamma}{\partial t} |S|^{3/2} d\mathbf{x} dt.$$

Proof. We prolongate S to a function \tilde{S} , defined on $[-T, 2T]$, by

$$\tilde{S}(t) := \begin{cases} S(-t), & \text{if } -T \leq t \leq 0, \\ S(t), & \text{if } 0 \leq t \leq T, \\ S(2T-t), & \text{if } T \leq t \leq 2T, \end{cases}$$

Owing to the corresponding properties of S , we can conclude that $\tilde{S} \in C([-T, 2T]; L^{3/2}(\Omega))$ and $(\partial/\partial t)\rho(\tilde{S}) \in L^{\infty}(-T, 2T; W^{-1,3}(\Omega))$. For $\Delta t > 0$ we define

$$X_{\Delta t} := \frac{1}{\Delta t} \int_0^T \int_{\Omega} \phi \left(\rho(\tilde{S}(t)) - \rho(\tilde{S}(t - \Delta t)) \right) \tilde{S}(t) d\mathbf{x}.$$

In the limit $\Delta t \rightarrow 0$ this expression tends to (cf. [7, proof of Lemma 1.2])

$$\lim_{\Delta t \rightarrow 0} X_{\Delta t} = \int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt.$$

Like in the proof of Lemma 3.1 an application of Young's inequality yields

$$\begin{aligned} X_{\Delta t} &\geq \frac{1}{\Delta t} \int_0^T \frac{1}{3} \int_{\Omega} \phi |\rho(\tilde{S}(t))| |\tilde{S}(t)| d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi |\rho(\tilde{S}(t - \Delta t))| |\tilde{S}(t - \Delta t)| d\mathbf{x} dt \\ &\quad + \frac{1}{\Delta t} \int_0^T \frac{2}{3} \int_{\Omega} (\phi \gamma(t) - \phi \gamma(t - \Delta t)) |\tilde{S}(t)|^{3/2} d\mathbf{x} dt \\ &= \frac{1}{3} \frac{\phi}{\Delta t} \left(\int_{T-\Delta t}^T \int_{\Omega} |\rho(\tilde{S}(t))| |\tilde{S}(t)| d\mathbf{x} dt - \int_{-\Delta t}^0 \int_{\Omega} |\rho(\tilde{S}(t))| |\tilde{S}(t)| d\mathbf{x} dt \right) \\ &\quad + \frac{2}{3} \int_0^T \int_{\Omega} \frac{\phi}{\Delta t} (\gamma(t) - \gamma(t - \Delta t)) |\tilde{S}(t)|^{3/2} d\mathbf{x} dt. \end{aligned}$$

Therefore we obtain in the limit $\Delta t \rightarrow 0$

$$\begin{aligned} \int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt &= \lim_{\Delta t \rightarrow 0} X_{\Delta t} \geq \frac{1}{3} \left(\int_{\Omega} \phi |\rho(S(T))| |S(T)| d\mathbf{x} - \int_{\Omega} \phi |\rho(S(0))| |S(0)| d\mathbf{x} \right) \\ &\quad + \frac{2}{3} \int_0^T \int_{\Omega} \phi \frac{\partial \gamma}{\partial t} |S(t)|^{3/2} d\mathbf{x} dt. \end{aligned}$$

Applying the same transformations and estimations to

$$Y_{\Delta t} := \frac{1}{\Delta t} \int_0^T \int_{\Omega} \phi \left(\rho(\tilde{S}(t + \Delta t)) - \rho(\tilde{S}(t)) \right) \tilde{S}(t) d\mathbf{x},$$

we find

$$\begin{aligned} \int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt &= \lim_{\Delta t \rightarrow 0} Y_{\Delta t} \leq \frac{1}{3} \left(\int_{\Omega} \phi |\rho(S(T))| |S(T)| d\mathbf{x} - \int_{\Omega} \phi |\rho(S(0))| |S(0)| d\mathbf{x} \right) \\ &\quad + \frac{2}{3} \int_0^T \int_{\Omega} \phi \frac{\partial \gamma}{\partial t} |S(t)|^{3/2} d\mathbf{x} dt. \end{aligned}$$

Together with the estimate from the consideration of $X_{\Delta t}$ above, the assertion follows. \square

Proposition 3.10. *The limits \mathbf{m} of $(\Pi_{\Delta t} \mathbf{m}_{\Delta t})_{\Delta t}$ and \mathbf{g} of $(\Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}))_{\Delta t}$ satisfy $\mathbf{g} = G(\mathbf{m})$, i.e., for all $\mathbf{v} \in L^1(0, T; (L^3(\Omega))^n)$:*

$$\int_0^T \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\mathbf{x} dt = \int_0^T \int_{\Omega} G(\mathbf{m}) \cdot \mathbf{v} d\mathbf{x} dt.$$

Proof. Again, we employ (27), replacing $a^k(\mathbf{m}^k, \mathbf{m}^k)$ by $\int_{\Omega} G^k(\mathbf{m}^k) \cdot \mathbf{m}^k d\mathbf{x}$, i.e.,

$$\int_{\Omega} G^k(\mathbf{m}^k) \cdot \mathbf{m}^k d\mathbf{x} + \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k d\mathbf{x} = \int_{\Omega} f^k S^k d\mathbf{x}$$

and the inequality (see the proof of Lemma 3.1)

$$\begin{aligned} \frac{1}{3} \int_{\Omega} \frac{\phi}{\Delta t} |\rho^k(S^k)| |S^k| d\mathbf{x} - \frac{1}{3} \int_{\Omega} \frac{\phi}{\Delta t} |\rho^{k-1}(S^{k-1})| |S^{k-1}| d\mathbf{x} \\ + \frac{2}{3} \int_{\Omega} \frac{\phi}{\Delta t} (\gamma^k - \gamma^{k-1}) |S^k|^{3/2} d\mathbf{x} \leq \int_{\Omega} \frac{\phi}{\Delta t} (\rho^k(S^k) - \rho^{k-1}(S^{k-1})) S^k d\mathbf{x}. \end{aligned}$$

Multiplying with Δt and summing up for $k = 1, \dots, K$, we obtain

$$\begin{aligned} \frac{1}{3} \int_{\Omega} \phi |\rho^K(S^K)| |S^K| d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi |\rho^0(S^0)| |S^0| d\mathbf{x} \\ + \frac{2}{3} \int_0^T \int_{\Omega} \phi (\Pi_{\Delta t} \gamma_{\Delta t}(t) - \Pi_{\Delta t} \gamma_{\Delta t}(t - \Delta t)) |\Pi_{\Delta t} S_{\Delta t}|^{3/2} d\mathbf{x} dt \\ + \int_0^T \int_{\Omega} \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) \cdot \Pi_{\Delta t} \mathbf{m}_{\Delta t} d\mathbf{x} dt \leq \int_0^T \int_{\Omega} \Pi_{\Delta t} f_{\Delta t} \Pi_{\Delta t} S_{\Delta t} d\mathbf{x} dt. \end{aligned}$$

Taking the limes inferior, we can conclude that

$$\begin{aligned} \frac{1}{3} \int_{\Omega} \phi |\rho(S(T))| |S(T)| d\mathbf{x} - \frac{1}{3} \int_{\Omega} \phi |\rho(S(0))| |S(0)| d\mathbf{x} \\ + \frac{2}{3} \int_0^T \int_{\Omega} \phi \frac{\partial \gamma}{\partial t} |S|^{3/2} d\mathbf{x} dt + \liminf_{\Delta t \rightarrow 0} \int_0^T \int_{\Omega} \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) \cdot \Pi_{\Delta t} \mathbf{m}_{\Delta t} d\mathbf{x} dt \leq \int_0^T \int_{\Omega} f S d\mathbf{x} dt. \end{aligned}$$

Thus Lemma 3.9 yields the following inequality

$$\int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt + \liminf_{\Delta t \rightarrow 0} \int_0^T \int_{\Omega} \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) \cdot \Pi_{\Delta t} \mathbf{m}_{\Delta t} d\mathbf{x} dt \leq \int_0^T \int_{\Omega} f S d\mathbf{x} dt.$$

On the other hand, Proposition 3.8 implies

$$\int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, S \right\rangle dt + \int_0^T \int_{\Omega} \mathbf{g} \cdot \mathbf{m} d\mathbf{x} dt = \int_0^T \int_{\Omega} f S d\mathbf{x} dt,$$

since from Proposition 3.7 c) and d) we have

$$\int_0^T \langle \operatorname{div}(\mathbf{m}), S \rangle dt = - \int_0^T \int_{\Omega} \mathbf{m} \cdot \nabla S d\mathbf{x} dt = \int_0^T \int_{\Omega} \mathbf{m} \cdot \mathbf{g} d\mathbf{x} dt .$$

Consequently,

$$\liminf_{\Delta t \rightarrow 0} \int_0^T \int_{\Omega} \Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) \cdot \Pi_{\Delta t} \mathbf{m}_{\Delta t} d\mathbf{x} dt \leq \int_0^T \int_{\Omega} \mathbf{g} \cdot \mathbf{m} d\mathbf{x} dt .$$

Thus we have shown that for arbitrary $\mathbf{v} \in L^\infty(0, T; (L^3(\Omega))^n)$

$$\int_0^T \int_{\Omega} (\mathbf{g} - G(\mathbf{v})) \cdot (\mathbf{m} - \mathbf{v}) d\mathbf{x} dt \geq \liminf_{\Delta t \rightarrow 0} \int_0^T \int_{\Omega} (\Pi_{\Delta t} G_{\Delta t}(\mathbf{m}_{\Delta t}) - \Pi_{\Delta t} G_{\Delta t}(\mathbf{v})) \cdot (\Pi_{\Delta t} \mathbf{m}_{\Delta t} - \mathbf{v}) d\mathbf{x} dt \geq 0 .$$

Now the assertion follows from the fact that G is a maximal monotone operator on $L^\infty(0, T; (L^3(\Omega))^n)$ (cf. the proof of Thm. 1.1 in [7]). \square

Now we are in a position to formulate and prove our main result:

Theorem 3.11. *For all $f \in L^\infty(0, T; L^3(\Omega))$ that are Lipschitz continuous in t there exists a pair $(\mathbf{m}, S) \in L^\infty(0, T; (L^3(\Omega))^n) \times L^\infty(0, T; W_0^{1,3/2}(\Omega))$ such that*

$$\begin{aligned} \int_0^T \int_{\Omega} G(\mathbf{m}) \cdot \mathbf{v} d\mathbf{x} dt - \int_0^T \langle \operatorname{div}(\mathbf{v}), S \rangle dt &= 0 & \text{for all } \mathbf{v} \in L^1(0, T; (L^3(\Omega))^n) , \\ \int_0^T \left\langle \phi \frac{\partial \rho(S)}{\partial t}, q \right\rangle dt + \int_0^T \langle \operatorname{div}(\mathbf{m}), q \rangle dt &= \int_0^T \int_{\Omega} f q d\mathbf{x} dt & \text{for all } q \in L^1(0, T; W_0^{1,3/2}(\Omega)) . \end{aligned}$$

Proof. Let \mathbf{m} be the limit of $(\Pi_{\Delta t} \mathbf{m}_{\Delta t})_{\Delta t}$ and S be the limit of $(\Pi_{\Delta t} S_{\Delta t})_{\Delta t}$. Then Proposition 3.10 and Proposition 3.7 d) imply that

$$\int_0^T \int_{\Omega} G(\mathbf{m}) \cdot \mathbf{v} d\mathbf{x} dt = \int_0^T \int_{\Omega} \mathbf{g} \cdot \mathbf{v} d\mathbf{x} dt = - \int_0^T \int_{\Omega} \nabla S \cdot \mathbf{v} d\mathbf{x} dt = \int_0^T \langle \operatorname{div}(\mathbf{v}), S \rangle dt$$

for all $\mathbf{v} \in L^1(0, T; (L^3(\Omega))^n)$. Thus (\mathbf{m}, S) satisfies the first equation above. In Proposition 3.8 we have seen that (\mathbf{m}, S) fulfills the second equation, too. \square

Remark 3.12. a) By means of the definition of the generalized divergence, we can replace the dual pairing $\langle \operatorname{div}(\mathbf{v}), q \rangle$ for $\mathbf{v} \in (L^3(\Omega))^n$ and $q \in W_0^{1,3/2}(\Omega)$ with the integral $-\int_{\Omega} \mathbf{v} \cdot \nabla q d\mathbf{x}$. Thus, in the case of the continuous transient problem, we have established the existence of a solution of the primal mixed formulation (cf. [8, Sect. I.3.2]). In contrast, we considered the uniqueness and existence of a solution of the dual mixed formulation for the stationary and semi-discrete transient Problem. This lack of regularity of the vector solution \mathbf{m} hinders the consideration of more general boundary conditions.

b) Assuming additional regularity properties of the solution, Amirat [2] showed that the solution to the corresponding parabolic Neumann-problem is unique. Furthermore, he proved that the solution is positive provided that the initial and boundary conditions satisfy corresponding requirements.

APPENDIX A. PROPERTIES OF $W^s(\operatorname{div}; \Omega)$

We introduce the generalization $W^s(\operatorname{div}; \Omega)$ of $H(\operatorname{div}; \Omega)$, defined by

$$W^s(\operatorname{div}; \Omega) := \{ \mathbf{v} \in (L^s(\Omega))^n \mid \operatorname{div}(\mathbf{v}) \in L^s(\Omega) \} ,$$

and equip it with the norm

$$\|\mathbf{v}\|_{W^s(\text{div};\Omega)} := \left(\int_{\Omega} \sum_{i=1}^n |v_i(\mathbf{x})|^s d\mathbf{x} + \int_{\Omega} |\text{div}(\mathbf{v}(\mathbf{x}))|^s d\mathbf{x} \right)^{1/s},$$

where $\mathbf{v} = (v_1, \dots, v_n)^T$. Since $W^s(\text{div};\Omega)$ is a closed subspace of $(L^s(\Omega))^{n+1}$, it follows that $W^s(\text{div};\Omega)$ is a reflexive Banach space.

It is straightforward to extend the proofs of Thm. 2.4 and Thm. 2.5 in [5] to show the next two lemmas:

Lemma A.1. *The space $\mathcal{D}(\bar{\Omega})^n$ is dense in $W^s(\text{div};\Omega)$.*

Lemma A.2. *The mapping $\gamma_n : \mathbf{v} \mapsto \mathbf{v} \cdot \mathbf{n}$ defined on $\mathcal{D}(\bar{\Omega})^n$ can be extended by continuity to a linear and continuous mapping, still denoted by γ_n , from $W^s(\text{div};\Omega)$ into $(W^{1/s,r}(\partial\Omega))'$. In particular, Green's formula*

$$\int_{\Omega} \mathbf{v} \cdot \nabla \psi d\mathbf{x} + \int_{\Omega} \text{div}(\mathbf{v}) \psi d\mathbf{x} = \int_{\partial\Omega} \psi (\mathbf{v} \cdot \mathbf{n}) d\sigma \quad (37)$$

holds for every $\mathbf{v} \in W^s(\text{div};\Omega)$ and $\psi \in W^{1,r}(\Omega)$, where $1/s + 1/r = 1$.

For $s > 1$, the well known inf-sup condition (see e.g. [3, §II.1]) can be extended, too. Generalizing the definition of the bilinear form b from Section 1 onto $W^s(\text{div};\Omega) \times L^r(\Omega)$, we define $b(\mathbf{v}, q) := \int_{\Omega} \text{div}(\mathbf{v}) q d\mathbf{x}$ for $\mathbf{v} \in W^s(\text{div};\Omega)$, $q \in L^r(\Omega)$.

Lemma A.3. *Let $s > 1$ and $1/s + 1/r = 1$. Then there exists a constant $\theta > 0$ such that*

$$\theta \|q\|_{0,r,\Omega} \leq \sup_{\mathbf{v} \in W^s(\text{div};\Omega)} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{W^s(\text{div};\Omega)}} \quad \text{for all } \mathbf{v} \in W^s(\text{div};\Omega), q \in L^r(\Omega). \quad (38)$$

Proof. We define a mapping $B : W^s(\text{div};\Omega) \rightarrow L^s(\Omega) = (L^r(\Omega))'$ by means of $\langle B\mathbf{v}, q \rangle = b(\mathbf{v}, q)$. Since $s > 1$, for every $p \in L^s(\Omega)$ there exists a Newtonian potential $N_p \in W^{2,s}(\Omega)$ such that $\Delta N_p = p$ almost everywhere. Then $\mathbf{v} := \nabla N_p$ satisfies $\mathbf{v} \in W^s(\text{div};\Omega)$ and $\text{div}(\mathbf{v}) = \Delta N_p = p$ in $L^s(\Omega)$. Therefore B is onto. Applying Lemma A.1 of [9], the assertion follows. \square

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